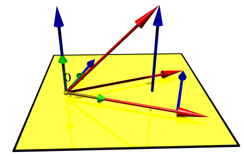
LINEAR ALGEBRA

MATH 21B



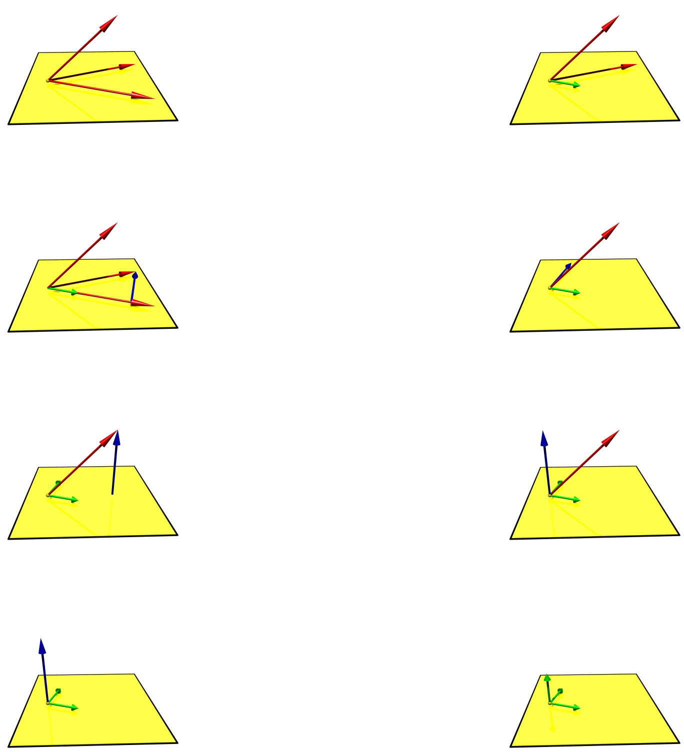
Gram Schmidt

13.1. We have seen that in order to project onto a space V, it is useful to have an **orthonormal basis** in V. The **Gram-Schmidt process** generates from a general basis $(\vec{v}_1, \dots, \vec{v}_n)$ an orthonormal basis $(\vec{u}_1, \dots, \vec{u}_n)$. The algorithm does the natural thing. It straightens up the vectors using projections and scales them to have length one. In order to find vectors perpendicular to the already straightened out space, we need projections because $\vec{v} - \text{proj}_V \vec{v}$ is perpendicular to V. Remember that if $\vec{u}_1, \dots, \vec{u}_k$ are an orthonormal basis in V, then

$$\text{proj}_V \vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_k \vec{u}_k$$

with $c_j = \vec{v} \cdot \vec{u}_j$.

13.2. First normalize $\vec{u}_1 = \vec{v}_1 / ||\vec{v}_1||$. The Gram-Schmidt process recursively constructs from the already constructed orthonormal set $\vec{u}_1, \dots, \vec{u}_{i-1}$ which spans a linear space V_{i-1} the new vector $\vec{w}_i = (\vec{v}_i - \text{proj}_{V_{i-1}}(\vec{v}_i))$ which is orthogonal to V_{i-1}, and then normalizes $\vec{w}_i$ to get $\vec{u}_i = \vec{w}_i / ||\vec{w}_i||$. Each vector $\vec{w}_i$ is orthogonal to the linear space V_{i-1}. The vectors $\{\vec{u}_1, \dots, \vec{u}_n\ \}$ form an orthonormal basis in V.



13.3. Find an orthonormal basis for $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

1. $\vec{u}_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
2. $\vec{w}_2 = (\vec{v}_2 - \text{proj}_{V_1}(\vec{v}_2)) = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \frac{\vec{w}_2}{||\vec{w}_2||} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
3. $\vec{w}_3 = (\vec{v}_3 - \text{proj}_{V_2}(\vec{v}_3)) = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$, $\vec{u}_3 = \frac{\vec{w}_3}{||\vec{w}_3||} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

13.4. This process also shows that any matrix A with linearly independent columns $\vec{v}_i$ can be decomposed as $A = QR$, where Q has orthonormal column vectors and where R is an upper triangular square matrix. The matrix Q has the orthonormal vectors $\vec{u}_i$ in the columns.

13.5. The matrix with the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$. In this case, the QR decomposition is easy: $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

13.6. The recursive formulae of the process were stated by Erhard Schmidt (1876-1959) in 1907. But the essence of the formulae were already in a 1883 paper of J.P.Gram in 1883 which Schmidt mentions in a footnote. It seems also that already Laplace (1749-1827) and Cauchy (1789-1857) in 1836 used the process.


Gram


Schmidt


Laplace


Cauchy

13.7. Something about the **transpose operation**. Some cool rules
(i) $(A^T)^T = A$
(ii) $(AB)^T = B^T A^T$
(iii) If A is arbitrary, then $A^T A$ and AA^T are both defined.
(iv) The columns of A are orthonormal if and only if $A^T A = 1$.
(v) If $A^T A = 1$, then AA^T is the projection onto the image of A. (Cute)