LINEAR ALGEBRA

MATH 21B

Determinants

14.1. The inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ was given by

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad - bc}$$

The expression $\det(A) = ad - bc$ is called the **determinant** of A.

14.2. Because determinants have emerged when solving linear equations, determinants have been around before matrices were introduced. You might have seen in the context of multivariable calculus the determinant of a 3×3 matrix because it represents the signed volume $\det(A) = \vec{v_1} \cdot (\vec{v_2} \times \vec{v_3}$ of the parallel-epiped spanned by the three column vectors. The sign is positive if the vectors form a right handed system and is negative if the vectors for a left handed system. The formula

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - fha - bdi$$

is **Sarrus rule**.

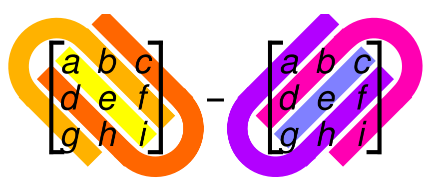

Figure 1. The Sarrus rule for the determinant of a 3×3 matrix.

14.3.

> **Definition:** The determinant of a $n \times n$ matrix sums over all patterns π of an $n \times n$ grid. The signature $\text{sign}(\pi)$ is 1 for an even number of up-crossings and -1 else.
>
> $$\det(A) = \sum_{\pi} \text{sign}(\pi) a_{1\pi(1)} \cdots (-1)^{n+1} a_{n\pi(n)} .$$

This is the **Leibniz definition** of determinants. One sees from this definition immediately that if a row or column is multiplied by a constant k the determinant gets multiplied by that constant and that if two rows or columns are switched, then the determinant changes sign. One can also see from the definition that $\det(A^T) = \det(A)$ because the pattern and a transposed pattern have the same signature.

14.4. There is a recursive way to compute the determinant. Look at the determinant of the matrix $A_{i,j}$ by deleting the i'th row and j'th column. Lets look at the first column and look at all patterns satisfying $\pi(1) = 1$. Summing over all these patterns gives $a_{11}\det(A_{1,1})$. Now look at all patterns satisfying $\pi(1) = 2$. There is one upcrossing coming from the first entry. Summing over all these patterns gives now $-a_{21}\det(A_{2,1})$. Continue as such and adding up gives

Theorem: $\det(A) = a_{11}\det(A_{1,1}) - a_{21}\det(A_{2,1}) + \cdots + a_{n1}\det(A_{n,1})$

One can do the same **Laplace expansion** for any row and any column. Just think of the matrix as a checkerboard. Assign the number 1 to every field where $i + j$ is even and -1 else. When you cross off the i'th row and j'th column multiply the minor with $(-1)^{i+j}a_{ij}$.

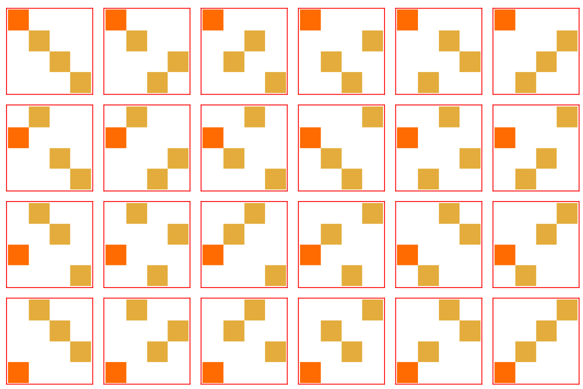

FIGURE 2. The Laplace expansion in the 4×4 case. The first 6 patters are $a_{11}\det(A_{1,1})$. The second is counted negative $-a_{21}\det(A_{2,1})$, then we have $a_{31}\det(A_{3,1})$ and finally $-a_{41}\det(A_{4,1})$.

14.5. One can immediately see from the Leibniz definition that one can compute the determinant fast using row reduction. In the homework, you derive this from the Laplace expansion formula.

Theorem: If two rows are switched, the sign of the determinant changes. If a row is multiplied by k, the determinant is multiplied by k.

14.6. An other consequence of the definition is (you also think about this in the homework):

Theorem: If A is upper triangular, the determinant is the product of the diagonal entries. Especially: $\det(1) = 1$.

14.7. Using row reduction, one can show that

Theorem: $\det(AB) = \det(A)\det(B)$ and so $\det(A^{-1}) = 1/\det(A)$ for an invertible matrix. Invertibility is equivalent to $\det(A) \neq 0$.