

LINEAR ALGEBRA

MATH 21B

CHECKLIST

- $n \times m$ **matrix** has m columns and n rows, maps domain \mathbf{R}^m to codomain \mathbf{R}^n .
- Square matrix**: $n \times n$ matrix, maps \mathbf{R}^n to \mathbf{R}^n .
- Identity matrix**: diagonal matrix I_n satisfying $I_n v = v$ for all vectors v .
- Column vector**: $n \times 1$ matrix, **Row vector**: a $1 \times n$ matrix
- Linear trafo**: $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$, $\vec{x} \mapsto A\vec{x}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$, $T(k\vec{x}) = kT(\vec{x})$.
- Column vectors of A** : images of **standard basis vectors** $\vec{e}_1, \dots, \vec{e}_n$.
- Linear system of equations**: $A\vec{x} = \vec{b}$, have n equations, m unknowns.
- Consistent system** $A\vec{x} = \vec{b}$: there is at least one solution \vec{x} .
- Vector form of linear equation**: $x_1\vec{v}_1 + \dots + x_n\vec{v}_m = \vec{b}$, \vec{v}_i columns of A .
- Matrix form of linear equation**: $\vec{w}_i \cdot \vec{x} = b_i$, \vec{w}_i rows of A .
- Augmented matrix of $A\vec{x} = \vec{b}$** : the matrix $[A|\vec{b}]$ which has one column more as A .
- Coefficient matrix of the system $A\vec{x} = \vec{b}$** : is the matrix A .
- Multiplication**: $[AB]_{ij} = \sum_k A_{ik}B_{kj}$, dot product row i of A with column j of B .
- Row reduction**: $A \rightarrow \text{rref}(A)$ in row reduced echelon form.
- Steps**: Swapping rows, Scaling row, Subtracting row from another row.
- Row reduced**: all nonzero rows have **leading 1**, columns with leading 1 are clean, every row above leading row has leading 1 to the left.
- Leading column**: column with leading 1 in $\text{rref}(A)$.
- Free variable**: a variable for which we have no leading 1 in $\text{rref}(A)$.
- Rank of matrix A** : number of leading 1 in $\text{rref}(A)$. Is equal $\dim(\text{im}(A))$.
- Nullity of matrix A** : the number of free variables. It is $\dim(\ker(A))$.
- Kernel of matrix**: $\{\vec{x} \in \mathbf{R}^m, A\vec{x} = \vec{0}\}$.
- Image of matrix**: $\{A\vec{x}, \vec{x} \in \mathbf{R}^m\}$.
- Inverse matrix of A** : matrix $B = A^{-1}$ satisfies $AB = BA = I_n$
- Rotation**: $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$, counterclockwise rotation by angle α .
- Dilation**: $\vec{x} \mapsto k\vec{x}$, also called scaling. Given by diagonal $A = kI_2$.
- Rotation-Dilation**: $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
- Reflection-Dilation**: $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

- **Shears:** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.
- **Line Reflection:** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$.
- **Line Projection:** $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ with $|\vec{u}| = 1$. Matrix $A = \begin{bmatrix} a^2 & ab \\ ba & b^2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.
- **Strange matrix:** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ satisfies $A^2 = 0$. $\text{Ker}(A) = \text{Im}(A)$.
- **Linear subspace:** check $\vec{0} \in V$, $\vec{x}, \vec{y} \in V, k \in \mathbf{R} \Rightarrow \vec{x} + \vec{y} \in V, k\vec{x} \in V$.
- **Orthogonality:** dot product $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$ is zero.
- **Length of a vector:** $\sqrt{\vec{v} \cdot \vec{v}}$.
- **Angle between vectors:** $\cos(\alpha) = \vec{v} \cdot \vec{w} / (|\vec{v}||\vec{w}|)$ is in $[0, \pi]$.
- $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ **span** V : every $\vec{x} \in V$ can be written as $\vec{x} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.
- $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ **linear independent** V : $a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}$ implies $a_j = 0$ all j .
- $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ **basis**: linear independent + span V . $V = \{\vec{0}\}$ has empty basis.
- **Dimension of linear space** V : number of basis elements of a basis in V .
- **S -matrix** : The matrix which contains the basis vectors as columns.
- **\mathcal{B} coordinates:** $c = S^{-1}\vec{v}$, where S is S -matrix.
- **\mathcal{B} matrix:** of T in basis \mathcal{B} . The matrix is $B = S^{-1}AS$ has columns $[T(\vec{v}_k)]_{\mathcal{B}}$.
- **A similar to B :** defined as $B = S^{-1}AS$. Does the same but in other coordinates.
- **Solutions:** a linear system of equations has either exactly 0, 1 or ∞ many solutions.
- **Solve system:** row reduce $[A|b]$ and parametrize with free variables.
- **Vectors perpendicular** V : get kernel of matrix which contains basis of V as rows.
- **Rank-nullity theorem:** $\dim(\ker(A)) + \dim(\text{im}(A)) = n$, where A is $m \times n$ matrix.
- **Kernel under elimination:** kernel stays invariant under Gauss-Jordan elimination.
- **Image under elimination:** image can change during Gauss-Jordan elimination.
- **Number of basis elements:** is independent of basis. Allows to define dimension.
- **Basis of $\text{im}(A)$:** pivot columns of A form a basis of the image of A .
- **Basis of $\ker(A)$:** introduce free variables for each redundant column of A .
- **Basis of of complement V^\perp of V :** this is a kernel computation.
- **Inverse of 2×2 matrix:** switch diagonal, negate wings and divide by determinant.
- **Inverse of $n \times n$ matrix:** Row reduce $[A|I_n]$ to get $[I_n|A^{-1}]$.
- **Kernel of product:** kernel of A is contained in the kernel of BA .
- **Image of product:** image of BA is contained in the image of B .
- **Matrix algebra:** $(AB)^{-1} = B^{-1}A^{-1}$, $A(B + C) = AB + AC$, $AB \neq BA$ in general.
- **A invertible:** $\Leftrightarrow \text{rref}(A) = I_n \Leftrightarrow \ker(A) = \{\vec{0}\} \Leftrightarrow \text{rank}(A) = n \Leftrightarrow \text{nullity}(A) = 0$.