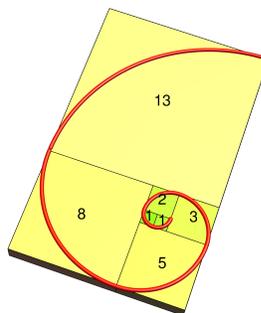


# LINEAR ALGEBRA

MATH 21B

## DISCRETE DYNAMICAL SYSTEMS



### 16.1. The Fibonacci recursion

$$F(t+1) = F(t) + F(t-1)$$

with  $F(0) = 0, F(1) = 1$  leads to the sequence  $0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$ . You always add up the previous 2 numbers. These numbers can be spotted in nature like sunflowers or pine cones or in drawings like the Fibonacci spiral. We can use linear algebra to describe this sequence by looking at the vector  $\vec{x}(t) = \begin{bmatrix} F(t+1) \\ F(t) \end{bmatrix}$  and see that

$$\vec{x}(t) = \begin{bmatrix} F(t+1) \\ F(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F(t) \\ F(t-1) \end{bmatrix} = A\vec{x}(t-1).$$

**16.2.** If we had a vector  $\vec{v}$  for which  $A\vec{v} = \lambda\vec{v}$ , we can get the time evolution because  $A^t\vec{v} = \lambda^t\vec{v}$ . If we had a basis  $\vec{v}_1$  and  $\vec{v}_2$  with  $A^t\vec{v}_1 = \lambda_1^t\vec{v}_1$  and  $A^t\vec{v}_2 = \lambda_2^t\vec{v}_2$  then the solution to any  $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$  could be written as

$$A^t\vec{v} = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2.$$

This is called the **closed form solution** of the problem. We will get back to Fibonacci later and also discuss in the next lectures how to get these eigenvalues and eigenvectors.

### 16.3.

**Definition:** A nonzero vector  $v$  is called an **eigenvector** of a  $(n \times n)$ -matrix  $A$  with **eigenvalue**  $\lambda$  if  $A\vec{v} = \lambda\vec{v}$ .

**Definition:** A basis  $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$  is called an **eigenbasis**, if each of the vectors is an eigenvector, meaning  $A\vec{v}_j = \lambda_j\vec{v}_j$ .

**16.4.** We will learn soon how to get eigenvalues and eigenvectors. For now, we assume them for granted.

**16.5.** Let us modify the above problem and assume to have a recursion

$$G(t+1) = G(t) + 2G(t-1)$$

with the same initial conditions  $G(0) = 0, G(1) = 1$ . The sequence

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \dots$$

can now be described by the discrete dynamical system

$$\vec{x}(t) = \begin{bmatrix} G(t+1) \\ G(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G(t) \\ G(t-1) \end{bmatrix} = A\vec{x}(t-1).$$

This is a situation, where the eigenvectors and eigenvalues of the matrix  $A$  are simpler. Just check that

$$A\vec{v}_1 = A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda_1 \vec{v}_1 .$$

$$A\vec{v}_2 = A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \vec{v}_2 .$$

We also have

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

Therefore

$$\vec{x}(t) = \frac{1}{3} 2^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{3} (-1)^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

We see that

$$G(t) = 2^t/3 - (-1)^t/3 .$$

For example,  $G(10) = 2^{10}/3 - 1/3 = 341$ . Isn't it amazing that we get an explicit formula for the  $t$ 'th term?

**16.6.** Let us go back to the Fibonacci case, where  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ . An important role there plays the **golden ratio**  $\phi = (1 + \sqrt{5})/2$ . You can check that

$$A\vec{v}_1 = A \begin{bmatrix} \phi \\ 1 \end{bmatrix} = \phi \begin{bmatrix} \phi \\ 1 \end{bmatrix} = \lambda_1 \vec{v}_1$$

and

$$A\vec{v}_2 = A \begin{bmatrix} 1 - \phi \\ 1 \end{bmatrix} = (1 - \phi) \begin{bmatrix} 1 - \phi \\ 1 \end{bmatrix} = \lambda_2 \vec{v}_2$$

And then

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (\vec{v}_1 - \vec{v}_2)/\sqrt{5}$$

Therefore

$$\vec{x}_n = \frac{1}{\sqrt{5}} \phi^n \begin{bmatrix} \phi \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} (1 - \phi)^n \begin{bmatrix} 1 - \phi \\ 1 \end{bmatrix} .$$

We see that

$$F(t) = ((1 + \sqrt{5})^n - (1 - \sqrt{5})^n)/(2^n \sqrt{5}) .$$

This is known as the **Binet formula**.

**16.7.** The golden mean  $\phi$  was named so 1835 by Luca Pacioli. It has also been called the **“divine proportion.”** The number appeared first in Euclid's elements around 350 BC. We can verify from the **formula of Binet** that

$$\frac{F(t+1)}{F(t)} \rightarrow \phi .$$

The Fibonacci numbers can be used to approximate the **golden ratio**.

**16.8.** We can use discrete dynamical systems also in geometric situation. If  $A$  is a reflection. What can you say about **the orbit**  $A^t$ ?