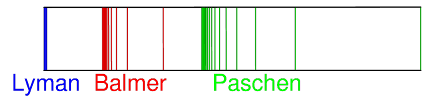
LINEAR ALGEBRA



MATH 21B

Eigensystem

17.1. Let A be a square matrix.

> **Definition:** A non-zero vector $\vec{v}$ is called an **eigenvector** of A, if $\boxed{A\vec{v} = \lambda\vec{v}}$. The number λ is called **eigenvalue**.

We ask $\vec{v}$ to be non-zero because otherwise, every λ would satisfy $A\vec{0} = \lambda\vec{0}$.

17.2. The matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ for example has an eigenvalue $\lambda_1 = 5$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and an eigenvalue $\lambda_2 = -2$ with eigenvector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

17.3. By rewriting the eigenvalue equation

$$(A - \lambda I)\vec{v} = 0 \ .$$

we see that $\vec{v}$ is a non-zero vector in $\ker(A - \lambda I)$.

> **Definition:** $\boxed{E_\lambda = \ker(A - \lambda I)}$ the called **eigenspace** of λ.

> **Definition:** $\boxed{f_A(\lambda) = \det(A - \lambda I)}$ is called **characteristic polynomial**.

If λ_k is a root of $f_A(\lambda)$, then there are non-zero vectors $\vec{v}_k$ such that $A\vec{v}_k = \lambda_k\vec{v}_k$ and E_λ is not trivial.

17.4. In the example case we have

$$f_A(\lambda) = \det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 10 \ .$$

Solving the quadratic equation $f_A(\lambda) = 0$ or factoring $f_A(\lambda) = (5 - \lambda)(-2 - \lambda)$ shows that $\lambda_1 = 5$ and $\lambda_2 = -2$ are eigenvalues.

17.5. To get the eigenvectors of an eigenvalue, just find the kernel of $A - \lambda I$. Here:

$$A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}, A + 2I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \ .$$

We see that $\mathcal{B} = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix})$ is an **eigenbasis**, a basis consisting of eigenvalues. The name "eigen" is German and stands for "self". It was David Hilbert, a German mathematician who introduced the term in 1904.

Definition: The multiplicity of an eigenvalue λ is called the **algebraic multiplicity** of λ. $\dim(E_\lambda)$ is the **geometric multiplicity** of λ.

The later is always smaller or equal than the algebraic multiplicity.

17.6. Example 1: If A is **diagonal** like

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} .$$

Then the standard basis is an eigenbasis as $A\vec{e}_1 = 3\vec{e}_1$ or $A\vec{e}_3 = 5\vec{e}_3$.

17.7. Example 2: If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation by 90 degrees, there is no real eigenvector. Indeed, the characteristic polynomial is $f_A(\lambda) = 1 + \lambda^2$ which has no real roots. The roots are complex $i, -i$. The corresponding eigenvectors are

$$\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix} .$$

17.8. Example 3: If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a shear matrix, then $\lambda_1 = 1$ and $\lambda_2 = 1$ are the eigenvalues. But

$$\ker(A - I) = \ker(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) .$$

We see that there is only one eigenvector to the two eigenvalues 1. There is no eigenbasis!

17.9. Example 4: $A = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$ is an example of a **stochastic matrix** or **Markov Matrix**. Every column is a probability vector. In this case there is always an eigenvalue 1. The reason is that the transpose matrix A^T has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix}$.

17.10. There is a matrix A whose eigenvalues and eigenvectors explains the periodic system of elements. This **Hydrogen atom matrix** has eigenvalues of the form $\lambda_n = \frac{1}{n^2}$. The differences between such eigenvalues are the energies which an atom can absorb or emit. The energy differences are organized as the **Lyman series** $(\frac{1}{1} - \frac{1}{m^2})$, **Balmer series** $(\frac{1}{4} - \frac{1}{m^2})$ and **Paschen series** $(\frac{1}{9} - \frac{1}{m^2})$. The **eigenspaces** of λ_n when organized a bit more, essentially determine the **atomic orbitals**. This should illustrate the importance of **eigenvalues** in our daily life: eigenvalues and eigenvectors determine the structure of the chemical elements we know in the universe. We are all made of eigenvectors!