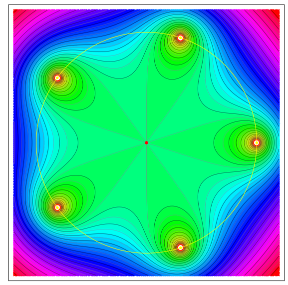

LINEAR ALGEBRA

MATH 21B

Diagonalization

18.1. If A is a square matrix with an **eigenbasis** $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$, the B matrix is $B = S^{-1}AS$, where the S matrix contains the vectors $\vec{v}_k$ as columns. This is a diagonal matrix because

$$S^{-1}AS\vec{e}_k = S^{-1}A\vec{v}_k = S^{-1}\lambda_k\vec{v}_k = \lambda_k S^{-1}\vec{v}_k = \lambda_k\vec{e}_k \ .$$

We call B the **diagonalization** of A. If S is real, it is a **diagonalization over the reals**. In general, it is a diagonalization over the complex numbers $\mathbb{C}$.

Definition: If there exists a matrix S such that $B = S^{-1}AS$ is diagonal, A is called **diagonalizable**.

Definition: If $B = S^{-1}AS$, then B is called **similar** to A.

Theorem: Similar matrices have the same eigenvalues.

The reason is that similar matrices have the same determinant and that if A, B are similar then $A - \lambda I$ and $B - \lambda I$ are similar too and are conjugated by same S. The characteristic polynomial $f_A(\lambda) = \det(A - \lambda I)$ therefore is the same than the characteristic polynomial $f_B(\lambda) = \det(B - \lambda I)$. So also the roots are the same.

18.2. We have seen that if a matrix has an eigenbasis, then it can be diagonalized. This can be reversed. If $B = S^{-1}AS$ is diagonal, just take the column vectors of S as the basis.

Theorem: A has an eigenbasis if and only if it can be diagonalized.

18.3. Which matrices can be diagonalized? Many matrices can be diagonalized but there are some like the shear matrix which can not. Some matrices like most rotations only can be diagonalized over the complex numbers.

Theorem: If all eigenvalues of A are distinct, then A can be diagonalized.

Proof: To see this assume that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$ with $A\vec{v}_k = \lambda_k v_k$, where λ_1 is the in absolute value largest eigenvalue. Now apply A^m to this equation. We have $c_1\lambda_1^m\vec{v}_1 + \cdots + c_n\lambda_n^m = 0$. Divide by λ_1^m to get $c_1 = -c_2\frac{\lambda_2^m}{\lambda_1^m}\vec{v}_2 - c_n\frac{\lambda_n^m}{\lambda_1^m} = 0$. As m goes to infinity, the right hand side goes to zero. Therefore $c_1 = 0$. Now do the same thing again for $c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$. Apply A^m and divide by λ_2^m again, we see $c_2 = 0$ etc. We get that $c_k = 0$ so that we have linear independence.

18.4. Example 1: Here is an other way to see discrete dynamical systems. Assume we want to find $A^{10^{100}}$ if $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. We can not iterate A so many times, but we can diagonalize A. We have $B = S^{-1}AS$ with $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Now $B^{10^{100}} = \begin{bmatrix} 2^{10^{100}} & 0 \\ 0 & 2^{10^{100}} \end{bmatrix}$.

18.5. Example 2: Diagonalization is useful if we want to apply functions on a matrix. Find a cube root of $A = \begin{bmatrix} 7 & 1 \\ -20 & 28 \end{bmatrix}$. In this case A is similar to $B = \begin{bmatrix} 27 & 0 \\ 0 & 8 \end{bmatrix}$ with $S = \begin{bmatrix} 1 & 1 \\ 20 & 1 \end{bmatrix}$. We have $B^{1/3} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Now $X = A^{1/3} = SB^{1/3}S^{-1} = \begin{bmatrix} 37 & 1 \\ -20 & 58 \end{bmatrix}/19$. How would you find the cube root X of A otherwise?

18.6. Example 3: The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a rotation in 5 dimensional space. It just permutes the standard basis. The characteristic polynomial is $f_A(\lambda) = 1 - \lambda^5$. The roots are $\lambda = 1^{1/5}$. Now in the complex, we can write $1 = e^{2\pi ik}$ for some integer k. We do this to get all 5 roots $1^{1/5} = e^{2\pi ik/5} = \cos(2\pi k/5) + i\sin(2\pi k/5)$. These 5 roots are located on a circle in the complex plane. Because these roots are different, we can find an eigenbasis. You can check that $\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{bmatrix}$ is an eigenvector to the eigenvalue λ. The corresponding matrix

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 & \lambda_5^3 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 & \lambda_4^4 & \lambda_5^4 \end{bmatrix}$$

diagonalizes A. What you just have seen is the **discrete Fourier transform** on a circular space with 5 elements. What we did here for 5 can be done for any $n \times n$ matrix which cyclically permutes the basis. Fourier theory and more generally harmonic analysis is all about diagonalization.