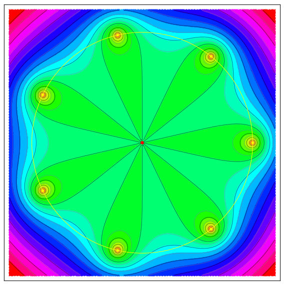
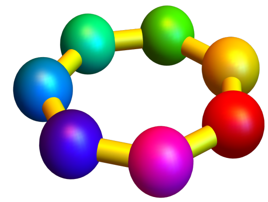


LINEAR ALGEBRA

MATH 21B



Complex Diagonalization

19.1. This is a continuation on **diagonalization**, especially in the case of complex eigenvalues. As a reminder, a matrix A is called **diagonalizable** if $B = S^{-1}AS$ is diagonal for some invertible matrix S. In that case, S contains an **eigenbasis** in the columns. Already for real matrices, it is possible that S is a complex matrix.

19.2. The prototype of a matrix which has complex eigenvalues is the **rotation-dilation matrix**

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The eigenvalues are $a + ib$ and $a - ib$. The eigenvectors are the columns of $S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. You can quickly check that $B = S^{-1}AS$. It is good to keep this example in mind!

19.3. Question. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$?

Solution. Do not compute when you know! A is a rotation-dilation matrix. The eigenvalues are $2 + i$ and $2 - i$ to the eigenvectors $\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

19.4. Here are some general rules which follow from looking at the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$. You can see for $\lambda = 0$ that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$ is the product of the eigenvalues.

> **Theorem:** $\det(A) = \prod_k \lambda_k$.

19.5. Also useful is that the **trace**, the sum of the diagonal elements stays the same. The reason is that it is a coefficient of the characteristic polynomial which does not change under a change of variables.

> **Theorem:** $\text{tr}(A) = \sum_k \lambda_k$.

19.6. We have seen that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the eigenvalues are the roots of the characteristic polynomial $\lambda^2 - T\lambda + D$, where $T = a + d$ is the trace of the matrix and $D = ad - bc$ are the eigenvalues of A. From the **quadratic formula** we see that $\lambda_k = \frac{T \pm \sqrt{T^2-4D}}{2}$. If $T^2 < 4D$ we have two complex conjugate eigenvalues. For $T^2 = 2D$ the two eigenvalues are the same.

19.7. Example: The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a rotation in 7 dimensional space. It just permutes the standard basis. The characteristic polynomial is $f_A(\lambda) = 1 - \lambda^7$. The roots are $\lambda = 1^{1/7}$. We can write $1 = e^{2\pi i k}$ for some integer k. Now take the 7'th root $1^{1/7} = e^{2\pi i k/7} = \cos(2\pi k/7) + i\sin(2\pi k/7)$. These 7 roots are located on a circle in the complex plane.

19.8. Because these roots are different, we can find an eigenbasis. The eigenvectors to the eigenvalue λ are $[1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \lambda^6]$ is an eigenvector to the eigenvalue λ. The corresponding matrix S containing all these eigenvectors is

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 & \lambda_7^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 & \lambda_5^3 & \lambda_6^3 & \lambda_7^3 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 & \lambda_4^4 & \lambda_5^4 & \lambda_6^4 & \lambda_7^4 \\ \lambda_1^5 & \lambda_2^5 & \lambda_3^5 & \lambda_4^5 & \lambda_5^5 & \lambda_6^5 & \lambda_7^5 \\ \lambda_1^6 & \lambda_2^7 & \lambda_3^7 & \lambda_4^7 & \lambda_5^7 & \lambda_6^7 & \lambda_7^7 \\ \lambda_1^7 & \lambda_2^7 & \lambda_3^7 & \lambda_4^7 & \lambda_5^7 & \lambda_6^7 & \lambda_7^5 \end{bmatrix}$$

It diagonalizes A.

19.9. Example: The matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

is the Kirchhoff matrix of a circular graph with 7 nodes. It is also called the Hamiltonian. Now, because we can write $L = A + A^T + 2I$ and $A^T = A^{-1}$ has the same eigenvectors with eigenvalue $\lambda_k^{-1} = \overline{\lambda}_k$, we can write know that L has the eigenvalues

$$\lambda_k = -e^{2\pi i k/7} - e^{-2\pi i k/7} + 2 = -2\cos(2\pi k/7) + 2 = 4\sin(\pi k/7) \ .$$

19.10. The matrix $B = S^{-1}LS$ is diagonal containing the eigenvalues λ_k in the diagonal. It might look strange that while S is complex, both L and B are real.