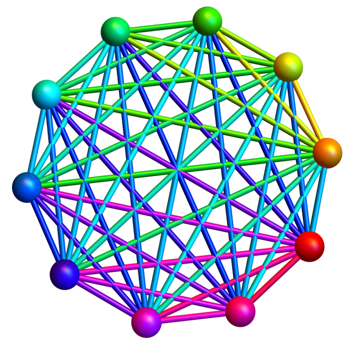


LINEAR ALGEBRA

MATH 21B

Spectral theorem

20.1. A matrix A is called **symmetric** or **self-adjoint** if $A^T = A$. In the case of a complex matrix, we ask $A^* = A$, where $A^* = \overline{A}^T$ is the **adjoint**.

20.2. A projection matrices $A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$ or reflection matrices $A = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ are symmetric. Shears like $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or rotations matrix $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ are not symmetric for or $\sin(\theta) \neq 0$ meaning $\theta \neq 0, \pi$.

20.3. The first part of the spectral theorem tells that symmetric matrices are "real":

Theorem: Symmetric matrices have real eigenvalues.

20.4. The proof needs to extend the dot product into the complex. Define $\langle \vec{v}, \vec{w} \rangle = \vec{v}^* \vec{w}$, $\vec{v}^*$ is the adjoint (when $\vec{v}$ is seen as a $n \times 1$ matrix). For example,

$$\langle \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix} \rangle = 1 + (-i)(-i) = 0 \; .$$

If the dot product is zero, the vectors are called **orthogonal**. As in $\mathbb{R}$, the number $|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ is the **length** or **magnitude** of $\vec{v}$. Since $|\vec{v}|^2 = |\vec{v}_1|^2 + \cdots + |\vec{v}_n|^2$, the length is zero if and only if $\vec{v} = \vec{0}$.

20.5. Here is now the proof that symmetric matrices have real eigenvalues. Let $A\vec{v} = \lambda \vec{v}$ and $A^* = A$. Now compute

$$\overline{\lambda} \langle \vec{v}, \vec{v} \rangle = \langle \lambda \vec{v}, \vec{v} \rangle = \langle A\vec{v}, \vec{v} \rangle = \langle A^* \vec{v}, \vec{v} \rangle = \langle \vec{v}, A\vec{v} \rangle = \langle \vec{v}, \lambda \vec{v} \rangle = \lambda \langle \vec{v}, \vec{v} \rangle$$

Because $\langle \vec{v}, \vec{v} \rangle > 0$ as eigenvectors can not be $\vec{0}$, we must have $\lambda = \overline{\lambda}$.

20.6. A matrix S is **orthogonal** if $S^T S = 1$. This means S has orthonormal columns.

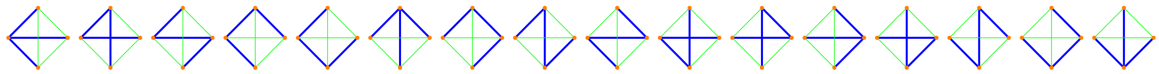
20.7. Here is the **spectral theorem**:

Theorem: A symmetric matrix is diagonalizable using orthogonal S.

Assume $A\vec{v} = \lambda \vec{v}$ and $A\vec{w} = \mu \vec{w}$ with $\lambda \neq \mu$ real.

$$\lambda \langle \vec{v}, \vec{w} \rangle = \langle \lambda \vec{v}, \vec{w} \rangle = \langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^* \vec{w} \rangle = \langle \vec{v}, A\vec{w} \rangle = \langle \vec{v}, \mu \vec{w} \rangle = \mu \langle \vec{v}, \vec{w} \rangle$$

shows that $(\lambda - \mu)\langle \vec{v}, \vec{w} \rangle = 0$ which means if $\lambda \neq \mu$ that $\langle \vec{v}, \vec{w} \rangle = 0$. As we can normalize the eigenvectors to have length 1, this proves the theorem in the case of simple spectrum. In general, just "wiggle" the matrix a bit, to unlock coincidences.



20.8. The most intuitive way to see why the spectral theorem holds in general is due to **Wigner and von Neumann**. They stated that every symmetric matrix A can be perturbed a bit within the class of symmetric matrices so that all eigenvalues are different. So, let $A_n \to A$ be a sequence of symmetric matrices which have simple spectrum. There are now orthogonal matrices such that $B_n = S_n^{-1} A_n S_n$ are diagonal. A sequence of orthogonal matrices has an accumulation point $S_{n_k} \to S$. Since multiplication is continuous and $A_{n_k} \to A$ we know that $B = S^{-1}AS$ is still diagonal.

20.9. Why does this wiggle argument not work for say $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$? We can still perturb A to have simple spectrum like $A_n = \begin{bmatrix} 1 & 1 \\ 0 & 1+1/n \end{bmatrix}$ which has eigenvalues $1+1/n$ and 1 and so can be diagonalized. What happens however is that the corresponding matrix S_n has the property that S_n^{-1} starts to diverge. This can not happen in the orthogonal case because $S_n^{-1} = S_n^T$ is bounded. Wiggling worked!

20.10. The spectral theorem can be extended to **normal matrices**, matrices which satisfy $A^*A = AA^*$. This includes orthogonal matrices. An eigenbasis is called **unitary** if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for $i \neq j$ and $\langle \vec{v}_i, \vec{v}_i \rangle = 1$. This is equivalent to the matrix S satisfying $S^*S = 1$. One calls such matrices **unitary**. A generalization of the spectral theorem:

Theorem: A is normal if and only if there is a unitary eigenbasis.

20.11. What happens if a matrix is not diagonalizable? It turns out that we can always **almost diagonalize it**. This is the content of the **Jordan normal theorem**.

Theorem: Any matrix can be brought into Jordan normal form.

20.12. Here is a matrix which appears in discrete mathematics. Take a network with $n = 10$ nodes in which all nodes are connected. Its Kirchhoff matrix is

$$L = \begin{bmatrix} 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 \end{bmatrix}$$

Since $U = A - 10I$ has everywhere an entry -1 we see that U has eigenvalues 0 with multiplicity 9 and -10 with multiplicity 1. The matrix A therefore has the eigenvalues 10 with multiplicity 9 and eigenvalue 0 with multiplicity 1. The matrix is diagonalizable because it is symmetric. The product of the non-zero eigenvalues is called the **Pseudo determinant** of L. It is 10^9. This is the number of rooted spanning trees in the network. The number of rooted trees is 10^8. In general for a complete network on n nodes, there are n^{n-2} spanning trees. This is the famous **Cayley tree formula**. For $n = 4$, there are $4^2 = 16$ spanning trees. Listed on the top of the page.