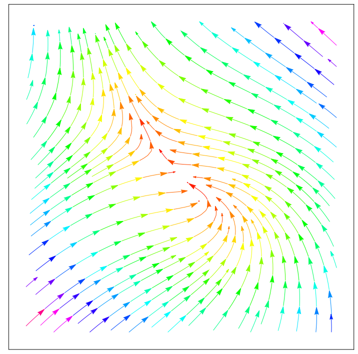

LINEAR ALGEBRA

MATH 21B

Differential equations

21.1. A **continuous dynamical system** $\boxed{\vec{x}\,'(t) = A\vec{x}(t)}$ is a continuum analog of a **discrete dynamical system** $\boxed{\vec{x}(t+1) = A\vec{x}(t)}$. Instead of deriving the state at time $t+1$ from the state at time t, we give the instantaneous change at the point. A discrete time example is ther **difference equation** $x(t+1) - x(t) = x(t)$ which allows to give $x(t+1) = 2x(t)$ and so get $x(t) = 2^t x(0)$. If we make the time intervals smaller like $x(t+h) - x(t) = hx(t)$ and take the limit $h \to 0$, we get the differential equation $x'(t) = x(t)$ which is an example of a **differential equation**. The solution is $x(t) = e^t x(0)$ as one can check by taking derivatives on both sides. We see again an exponential growth. Going from a discrete to an **instantaneous change** accelerated the growth from 2^t to e^t.

21.2. The **linear continuous dynamical systems** $\boxed{\vec{x}\,'(t) = A\vec{x}(t)}$ are a subclass of more general **ordinary differential equations** $\vec{x}\,'(t) = \vec{F}(\vec{x}(t))$, where $\vec{F}$ is a vector which a vector field, a vector which depends on $\vec{x}$. We sometimes write things down also as $(x\,'(t), y\,'(t)) = F(x(t), y(t))$. In the linear case like $(x\,'(t), y\,'(t) = (3x(t), -4y(t))$ we can immediately write down the solution $(x(t), y(t)) = (x(0)e^{3t}, y(0)e^{-4t})$. In vector form, this is written the system as

$$\begin{bmatrix} x\,'(t) \\ y\,'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} .$$

21.3. Like discrete dynamical systems $\vec{x}(t+1) = A\vec{x}(t)$, also continuous dynamical systems $\vec{x}\,'(t) = A\vec{x}(t)$ can be considered in higher dimension. **Diagonalization** decouples the system to one-dimensional systems. Consider therefore first the case $n = 1$, where $A = \lambda$ is a 1×1 matrix. It is useful to compare with the general one-dimensional discrete dynamical system

$$x(t+1) = \lambda x(t)$$

with the general one-dimensional continuous dynamical system

$$x\,'(t) = \lambda x(t) .$$

In both cases, we can write down the **closed form solution**.

21.4. In the discrete case, we have $\boxed{x(t) = \lambda^t x(0)}$. In the continuous case, we have $\boxed{x(t) = e^{\lambda t} x(0)}$. In either case, a system is called **asymptotically stable** or simply **stable** if $\vec{x}(t) \to \vec{0}$ for all initial conditions $\vec{x}(0)$. The condition for **stability** in the discrete case is $\boxed{|\lambda| < 1}$. In the continuous case the condition is $\boxed{\mathrm{Re}(\lambda) < 0}$. These solution formulas also work if λ is complex.

21.5. In the case of a 2×2 matrix, we can use eigenvalues and eigenvectors to bring the system into diagonal form. But rather than doing the diagonalization explicity, we can repeat what was done in the discrete. For example, for $A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$ and an initial condition like $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is given, we compute the eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ belonging to the eigenvalues $\lambda_1 = 6, \lambda_2 = 3$. The initial condition is written as a linear combination of these eigenvectors

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

After fixing these constants $c_1 = 10/3$ and $c_2 = 1/3$, the closed-form solution is

$$\vec{x}(t) = \frac{10}{3} e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3} e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

21.6. We have now a **closed form solution** which shows that asymptotic stability is equivalent with all eigenvalues λ_k have the property $\mathrm{Re}(\lambda_k) < 0$.

Theorem: If $\mathcal{B} = (\vec{v}_1, \cdots, \vec{v}_n)$ is an eigenbasis of A and $\vec{x}(0) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, then $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$ solves $\vec{x}\,'(t) = A\vec{x}(t)$.

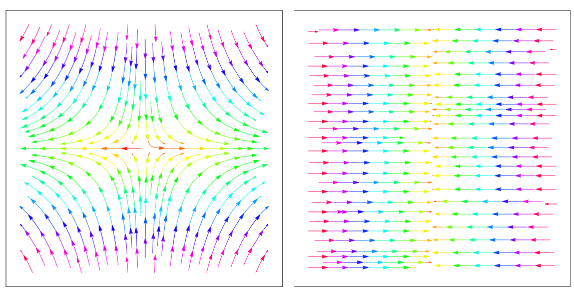

FIGURE 1. Trajectories of continuous dynamical systems with real eigenvalues. The first is $x' = -2x, y' = y$. The second is $x' = -y, y' = 0$.

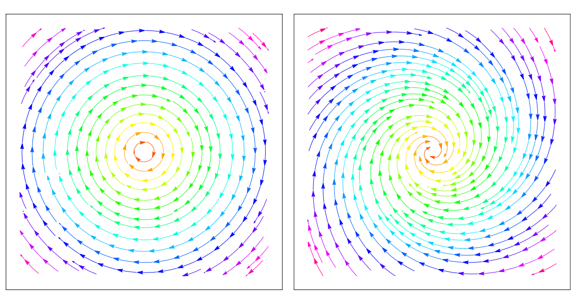

FIGURE 2. Trajectories of continuous dynamical systems with complex eigenvalues. The first is $x' = y, y' = -x$ the second is $x' = -x + 3y, y' = -3x - y$. The last is stable.