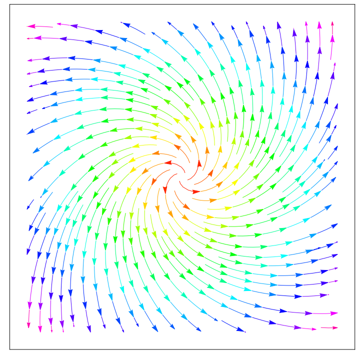

LINEAR ALGEBRA

MATH 21B

Matrix Exponential

22.1. We have seen that $\vec{x}\,'(t) = A\vec{x}(t)$ can be solved by a closed form solution: first write $\vec{x}(0) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ the initial condition in terms of $\vec{v}_1$ and $\vec{v}_2$, then write down the closed form solution

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n \ .$$

22.2. In the case of a 2×2 matrix A, there are different type of possibilities. Lets look first at the case of real eigenvalues and where we can diagonalize. Now we have 3 different possibilities A) $\lambda_1 < 0, \lambda_2 < 0$ B) $\lambda_1 > 0, \lambda_2 > 0$. C) $\lambda_1 < 0, \lambda_2 > 0$ with non-zero eigenvalues:

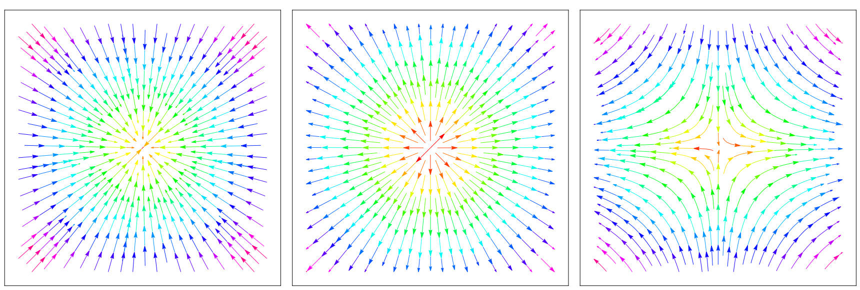

Figure 1.

22.3. Then there are 3 possibilities with a zero eigenvalue D) $\lambda_1 = 0, \lambda_2 > 0$, E) $\lambda_1 = 0, \lambda_2 < 0$. F) $\lambda_1 = 0, \lambda_2 = 0$.

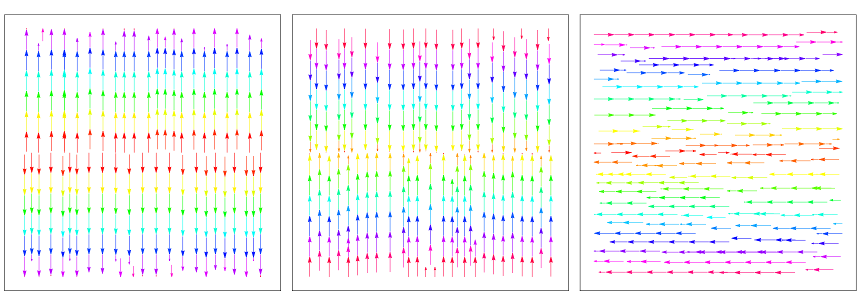

Figure 2.

22.4. In the case of complex eigenvalues, there eigenvalues come in pairs if the matrix A is real. There are now three possibilities only because if one eigenvalue is zero, the other is zero too. G) $\mathrm{Re}(\lambda_1) < 0$, H) $\mathrm{Re}(\lambda_1 = 0$ I) $\mathrm{Re}(\lambda_1) > 0$.

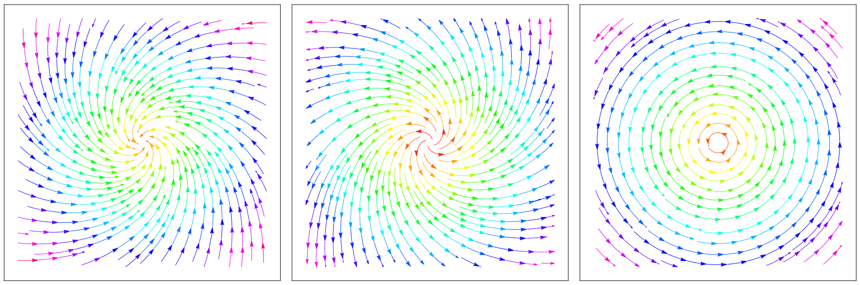

FIGURE 3.

22.5. What do we do if we do not have an eigenbasis? There is an other method: The differential equation $x' = \lambda x$ is solved by $e^{\lambda t}x(0)$. Why now just write in the matrix instead of λ and write

$$\vec{x}(t) = e^{At}x(0)$$

It actually can be verified quite quickly that this is a solution because if we differentiate

$$e^{At} = 1 + At + A^2t^2/2! + A^3t^33! + ...$$

with respect to t we get

$$0 + A + A^2 2t/2 + A^3 3t^2/3! + ... = A(1 + At + A^2t^2/2! + ...) = Ae^{At} .$$

So $\frac{d}{dt}e^{At}\vec{x}(0) = Ae^{At}\vec{x}(0)$.

Theorem: $\vec{x}'(t) = A\vec{x}(t)$ is solved by $\vec{x}(t) = e^{At}\vec{x}(0)$.

22.6. If $B = S^{-1}AS$ is diagonal, then $B^n = S^{-1}A^nS$ and $A^n = SB^nS^{-1}$ so that

$$e^{At} = Se^{Bt}S .$$

Diagonalization allows to compute functions of a matrix. We could even compute functions which are not differentiable and have no Taylor expansion. In general, one has a **functional calculus** for any function f:

Theorem: For diagonalizable A, we can define $f(A) = Sf(B)S^{-1}$.

22.7. Here is an example, where we can not use an eigenbasis. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Now $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and in general $A^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If we add up $1 + At + A^2t^2/2! + ...$ we get the matrix $I + tA = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. And indeed, we can check that $e^{At} = 1 + tA$. We see that $x(t) = x(0) + ty(0)$ and $y(t) = y(0)$ is the solution.

22.8. In the case $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, we have the eigenvalues $a + ib$. If b is non-zero, then the sign of a determines which of the cases in Figure 3) we deal with. We have **stability** if and only if $a < 0$. We talk about stability next time. A linear system is **stable** if $\vec{x}(t) \to \vec{0}$ for all initial conditions $\vec{x}(0)$.