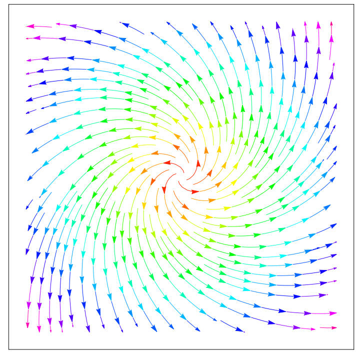

LINEAR ALGEBRA

MATH 21B

Stability

23.1. If $\vec{x}' = \vec{F}(\vec{x})$ is a differential equation and $\vec{v}$ is such that $\vec{F}(\vec{v}) = \vec{0}$, then it is called an **equilibrium point**. If we start at an equilibrium point, we stay there. The concept of equilibrium point will be especially interesting in the case of non-linear systems. In the case of linear systems we always have the origin $\vec{v} = \vec{0}$ as an equilibrium point.

23.2. When we look at the notion of stability, we mean **asymptotic stability**. The definition will also be used next time, when we look at non-linear systems.

> **Definition:** An equilibrium point $\vec{v}$ is called **stable** for the system $\vec{x}' = \vec{F}(\vec{x}(t))$ if for all $\vec{x}(t)$ close enough to $\vec{v}$, the solution $\vec{x}(t)$ converges to $\vec{v}$. In the case of a linear system, we say the system is stable if $\vec{0}$ is a stable equilibrium.

23.3. The one dimensional case leads the way. If we look at a one-dimensional linear system $\vec{x}'(t) = \lambda\vec{x}(t)$, then the solution is $\vec{x}(t) = e^{-\lambda t}\vec{x}(0)$. Stability is equivalent with $\lambda < 0$. In general:

> **Theorem:** A system $\vec{x}'(t) = A\vec{x}(t)$ is stable if and only if all eigenvalues λ_k of A have the property $\text{Re}(\lambda_k) < 0$.

23.4. Note that if $B = S^{-1}AS$, then $B^n = S^{-1}A^nS$ so that B is stable if and only if A is stable. A diagonal or upper triangular matrix produces a stable system if all diagonal entries are $\text{Im}(\lambda_k) \leq 0$. In the case when A is diagonalizable and $A = SB{+}S^{-1}$ with diagonal B containing the eigenvalues, then the stability statement is clear. In general, this is not so obvious. [One of the possibilities is to use the Schur triangulation theorem that assures that every matrix is conjugated to an upper triangular matrix using a complex matrix $SS^* = I$, then analyze the upper triangular case.]

23.5. Here are two examples of a stable system $\vec{x}'(t) = A\vec{x}$ with $A = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}$ and $\vec{x}'(t) = B\vec{x}$ with $B = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$. Both matrices have eigenvalues $-1 \pm 3i$. Can you figure out which is which ? As a preparation for next lecture, we drew also the **nullclines**. These are the lines where the field is horizontal or vertical. One of the things we will do on the nullclines is decide in which direction the vectors point there.

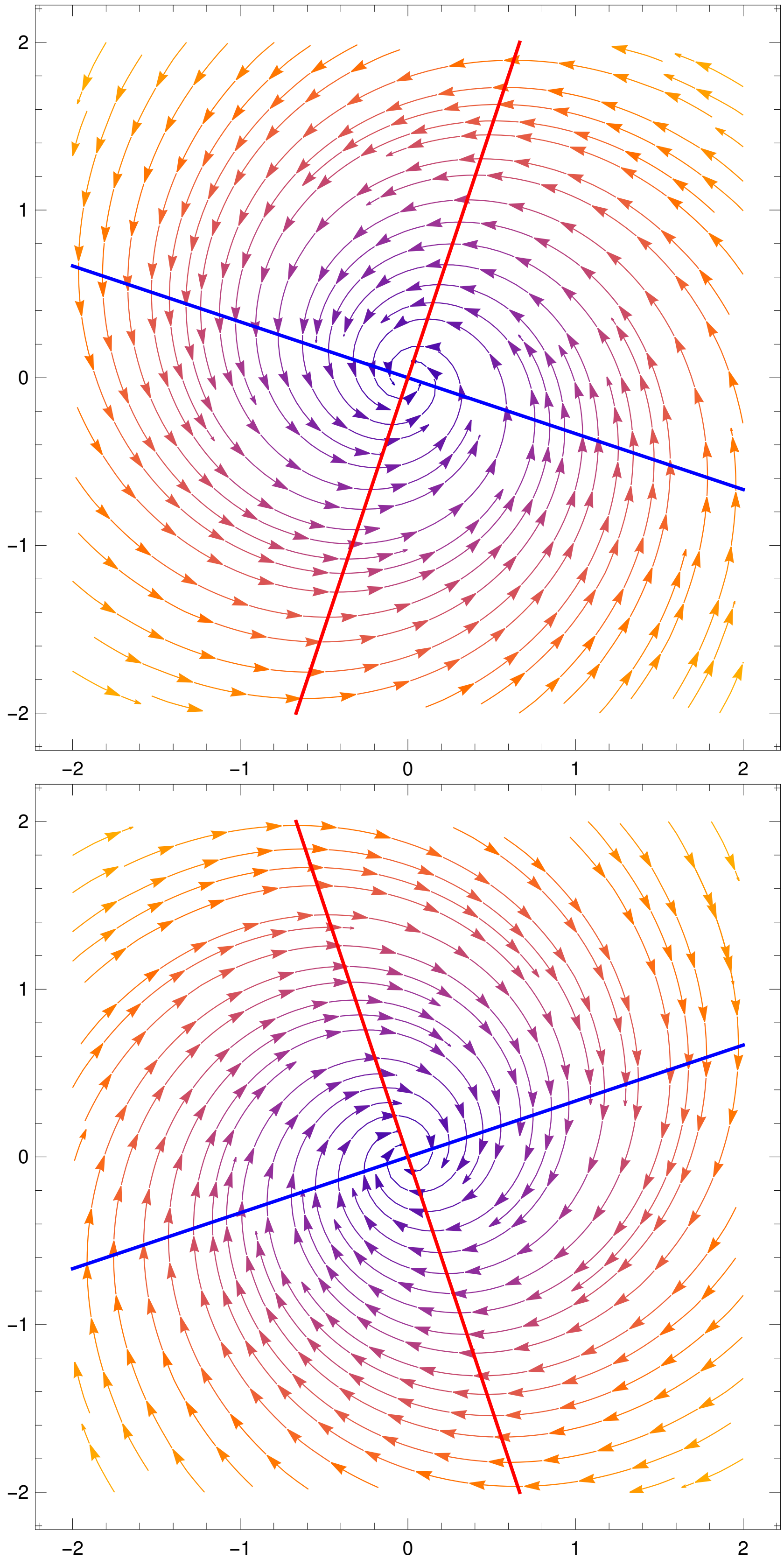

Figure 1.