LINEAR ALGEBRA

MATH 21B

NONLINEAR SYSTEMS
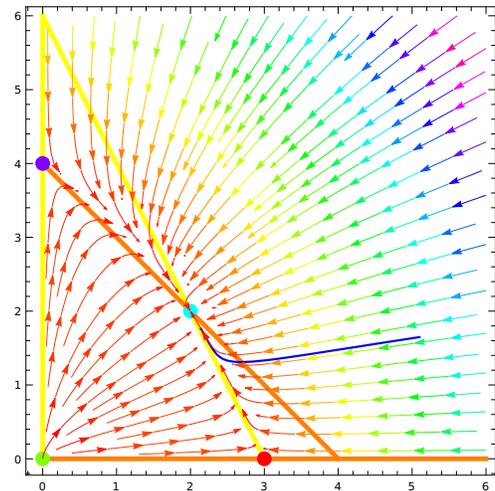
24.1. For a **differential equation**

$$\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right] = \left[\begin{array}{c} f(x,y) \\ g(x,y) \end{array}\right]$$

we talk about a **nonlinear system** if the functions $f(x,y)$ and $g(x,y)$ are nonlinear.

24.2. An example of a system in population dynamics is the **Murray system**

$$
\begin{aligned}
x' &= x(6 - 2x) - xy \\
y' &= y(4 - y) - xy \, .
\end{aligned}
$$

Without the xy interaction part, it consists of two independent **logistic systems**. The interaction is a **competition** because the other species produces a slow down (both have negative sign). With both having positive sign, we would have **symbiosis**. With one having positive and one negative, we would have a **predator-pray** situation.

24.3. A point (x,y) is called an **equilibrium point** if both $f(x,y) = 0$ and $g(x,y) = 0$. The x-**nullclines** are where $f(x,y) = 0$ and also for y-nullclines, where $g(x,y) = 0$. On x-nullclines, the vector field is vertical, while on y-nullclines, the vector field

> **Definition:** The **Jacobian matrix** at a point (x_0, y_0) is
> $$= \left[\begin{array}{cc} f_x & f_y \\ g_x & g_y \end{array}\right]$$
> the **linearization** near the point.

In multi-variable calculus one calls $[f_x, f_y]$ the gradient. At an equilibrium point (x_0, y_0) we have now a linearized system $\vec{x}'(t) = A\vec{x}(t)$. We call the equilibrium point **asymptotically stable** if its linearization is asymptotically stable.

24.4. To analyze a non-linear system, we find the nullclines, the equilibrium points, draw the directions on these nullclines, then linearize the system near each equilibrium point, then draw the phase portraits near the equilibrium points and finally connect the dots to see the global phase portrait. Let us do that in the case of the Murray system. Since $f(x,y) = x(6 - 2x - y)$, the x-nullclines consist of two lines $x = 0$ and $y = 6 - 2x$. Since $g(x,y) = y(4 - x - y)$ the y-nullclines are $y = 0$ and $y = 4 - x$. The equilibrium points are $(0,0), (3,0), (0,4), (2,2)$.

24.5. We can now analyze the system at each point:

Equilibrium	Jacobian	Eigenvalues	Type
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6,\ \lambda_2 = 4$	Unstable source
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6,\ \lambda_2 = 1$	Hyperbolic saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2,\ \lambda_2 = -4$	Hyperbolic saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink

24.6. The **Volterra-Lodka systems** is a population model that is more than 100 years old.

$$\dot{x} = 0.4x - 0.4xy$$
$$\dot{y} = -0.1y + 0.2xy$$

It has equilibrium points $(0,0)$ and $(1/2, 1)$. It describes a **predator-prey situation** like for example a **shrimp-shark population**, where (you guessed right), the shrimps are the pray. The shrimp population $x(t)$ becomes smaller with more sharks. The shark population on the other hand grows with more shrimp. Volterra explained with this model the oscillation of fish populations in the Mediterranean sea.
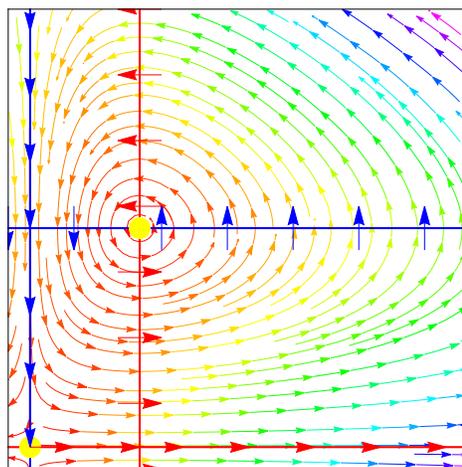

FIGURE 1.   The Volterra-Lodka **predator-prey** system. The Nullclines are all straight lines. After indicating the direction of the flow on a nullcline, the picture can be more easily completed.

24.7. The strategy to analyze a non-linear system follows a recipe:

1) Draw the null-clines and the direction of the field on these nullclines.
2) Identify the equilibria by intersecting different type of null-clines.
3) Find the eigenvalues and eigenvectors of the Jacobian and identify the type.
4) Draw the phase portrait near each equilibrium point
5) See how the trajectories fit together.