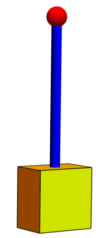
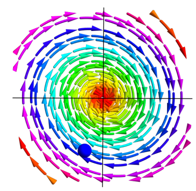


LINEAR ALGEBRA

MATH 21B



Second order differential equations

25.1. To solve a **second order differential equation**

$$f'' + bf' + cf = 0$$

with constants b, c, one can proceed in two ways. The first is to write an equivalent **first order system** and solve it with the methods we have learned.

25.2. To solve for example $f'' + 3f' + 2f = 0$, introduce $g = f'$ and write the system of equations

$$\begin{aligned} f' &= g \\ g' &= 3g - 2f \end{aligned}$$

which is the system $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}$. If you compute the characteristic polynomial, something interesting happens. It is $f_A(\lambda) = \lambda^2 + 3\lambda + 2$. In the general case, the characteristic polynomial is $\lambda^2 + b\lambda + c$.

25.3. After writing down the **closed form solution**, we see that

$$f(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} .$$

At least if $\lambda_1 \neq \lambda_2$. In the case when $\lambda_1 \neq \lambda_2$, we can not diagonalize but we can see that

$$f(t) = Ae^{\lambda_1 t} + Bte^{\lambda_2 t}$$

solves the system.

25.4. Here is the short cut:

Theorem: To solve $f'' + bf' + cf = 0$, find the roots λ_1, λ_2 of the characteristic polynomial $\lambda^2 + b\lambda + c = 0$. If $\lambda_1 \neq \lambda_2$, then write

$$f(t) = Ae^{t\lambda_1 t} + Be^{t\lambda_2 t}$$

is the general solution. If $\lambda_1 = \lambda_2$, the general solution is

$$f(t) = Ae^{t\lambda_1 t} + Bte^{t\lambda_1 t}$$

25.5. You can verify this by running the following two lines. We have set $a = \lambda_1$ and $b = \lambda_2$ and use $(\lambda - a)(\lambda - b) = \lambda^2 - (a+b)\lambda + ab$.

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f[t_]:=A Exp[t a] + B Exp[t b];      Simplify[f''[t]-(a+b) f'[t]+a*b f[t] ==0]
f[t_]:=A Exp[t a] + B t  Exp[t a];  Simplify[f''[t]-2a f'[t]+a^2 f[t] ==0]
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Alternatively, you can write the system as $(D^2 + bD + c)f = 0$, where $Df = f'$ is the derivative. Now we can factor as usual and say $(D - \lambda_1)(D - \lambda_2)f = 0$. we see that any f satisfying $(D - \lambda_1)f = 0$ solves the system. But this means $f(t) = Ae^{\lambda_1 t}$. Similarly any f satisfying $(D - \lambda_2)f = 0$ solves the system leading to $Be^{\lambda_2 t}$.

25.6. Example 1: Find the general solution of $f'' + 4f + 3 = 0$. The polynomial $\lambda^2 + 4\lambda + 3$ has the roots $\lambda_1 = -3, \lambda_2 = -1$. The general solution is $f(t) = Ae^{-3t} + Be^{-t}$.

25.7. Example 2: Find the general solution of $f'' - 6f + 9 = 0$. The polynomial $\lambda^2 - 6\lambda + 9$ has the roots $\lambda_1 = 3, \lambda_2 = 3$. The general solution is $f(t) = Ae^{3t} + Bte^{3t}$.

25.8. The theorem also works when the eigenvalues are complex. In that case it is even simpler as the two eigenvalues are then automatically disjoint.

25.9. Example 3: Solve $f'' + f = 0$. The roots of $\lambda^2 + 1 = 0$ are $\lambda_1 = i$ and $\lambda_2 = -i$. From the theorem we see that

$$f(t) = Ae^{it} + Be^{-it}$$

is the general solution of the differential equation. Because $e^{it} = \cos(t) + i\sin(t)$ and $e^{-it} = \cos(t) - i\sin(t)$ we can also use a different "basis" and note that $\text{span}(e^{it}, e^{-it})$ is $\text{span}(\cos(t), \sin(t))$ if we allow complex numbers.

25.10. Example 4: Solve $f'' - 1f' + (5/4)f = 0$. The roots of $\lambda^2 - \lambda + (5/4) = 0$ are $\lambda_1 = i + i$ and $\lambda_2 = 1 - i$. From the theorem we see that the general solution is

$$f(t) = Ae^{(1/2+i)t} + Be^{1/2-i)t} .$$

Again, we can replace this with $f(t) = Ae^{t/2}\cos(t) + Be^{t/2}\sin(t)$ which has the advantage that it is real. The solutions escape to infinity exponentially fast.

25.11. Example 5: Solve $f'' + 2f' + 2f = 0$. The roots of $\lambda^2 + 2\lambda + 2 = 0$ are $\lambda_1 = -i + i$ and $\lambda_2 = -1 - i$. From the theorem we see that the general solution is

$$f(t) = Ae^{(-1+i)t} + Be^{(-1-i)t} .$$

or $f(t) = Ae^{-t}\cos(t) + Be^{-t}\sin(t)$.

25.12. Example 6: Solve $f'' + 2f' = 0$. The roots of $\lambda^2 + 2\lambda = 0$ are $\lambda_1 = 0$ and $\lambda_2 = -2$. From the theorem we see that the general solution is

$$f(t) = A + Be^{-1t} .$$

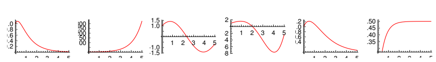

FIGURE 1. Solution curves for examples 1-6 with $f(0) = 1, f'(0) = 1$.