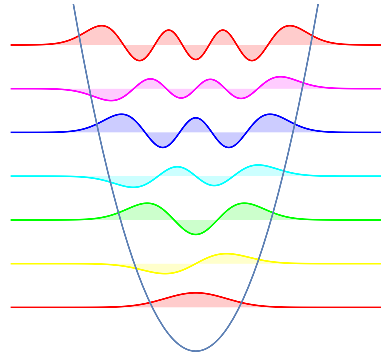

LINEAR ALGEBRA

MATH 21B

Linear Spaces

26.1. Linear spaces are sets in which one can add and multiply with scalars. Examples are **polynomials**, or **smooth functions** or **spaces of matrices**. The reason why we look at them as vectors is that we can add them, scale them and because there is a 0 element in them which when added, does not do anything. All notions we have learned like dimension, image, kernel, eigenvalues, linear dependent etc work also here.

Definition: A **linear space** V is a set of objects with an addition and scalar multiplication satisfying
(i) there is a 0 in V with the property $0 + f = f$ for all f.
(ii) If f, g are in V then $f + g$ is in V.
(iii) If f is in V and k is in $\mathbb{R}$ then kf is in V.

26.2. You see often the first condition omitted. We need to add it however as otherwise the empty set would count as a linear space. It is also useful, to check, before looking at other conditions that 0 is in V. We have seen this for linear subspaces already.

26.3. Here are examples of linear spaces:

- The space $\mathbb{R}^n$.
- A linear subspace of $\mathbb{R}^n$.
- The space $M(n, m) = \mathbb{R}^{n \times m}$ of all $n \times m$ matrices.
- The space P of all polynomials.
- The space P_n of all polynomials of degree $\leq n$.
- The space C^∞ of all smooth functions
- The space C of all continuous functions which satisfy $f(3) = 0$.
- The space $C^\infty(\mathbb{R}^2)$ of all smooth functions of two variables.

26.4. And here are example of spaces that are not linear spaces

- The space of all non-constant linear functions $f = ax + b$.
- The space of quadratic polynomials with exactly 2 roots.
- The space of all 2×2 matrices which are not invertible.
- The space of all smooth functions which have a root somewhere
- The space of continuous functions which satisfy $f(0) = 3$.
- The space of all 2×2 matrices which can be diagonalized
- The space of all reflection matrices.
- The space of all 2×2 matrices which satisfy $A^2 = 0$.

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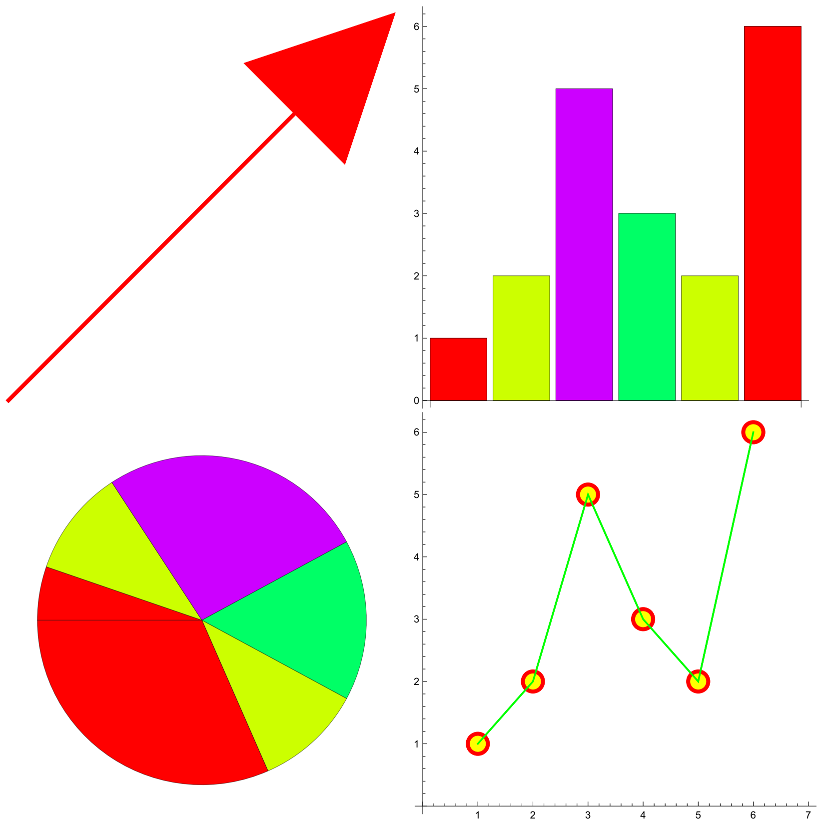

FIGURE 1. The vector $[1, 2, 5, 3, 2, 6]^T$ in $\mathbb{R}^6$ can be interpreted also as a function. Think of $f(k) = v_k$. The index k is the input and the output is the value of the function is v_k. By looking at the function, we can visualize it and also see it as data.

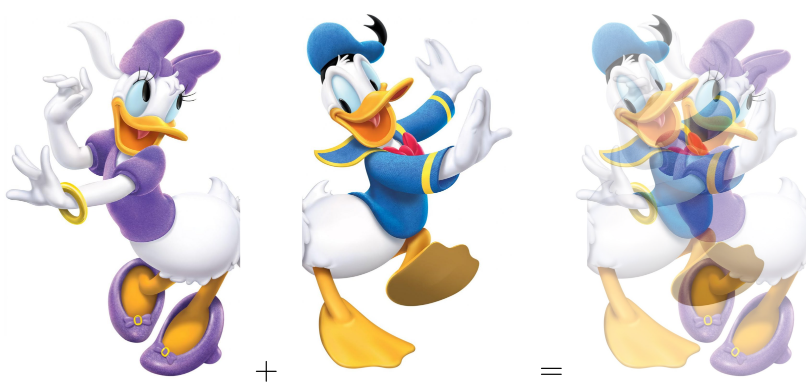

FIGURE 2. In a linear space we can add. Pictures can be seen as elements of a linear space. They can also be seen as a function. To every pixel (x, y) is assigned a color value $(r(x, y), g(x, y), b(x, y))$ telling how much red, green and blue there is at this point. Every color canal is a matrix. One can see a picture also as a vector consisting of 3 matrices with size $w \times h$. Technically, a picture is an element in $\mathbf{R}^{w,h} \times \mathbf{R}^{w,h} \times \mathbf{R}^{w,h}$. Our Donald and Daisy pictures had $w = 1000, h = 1544$ width and height.