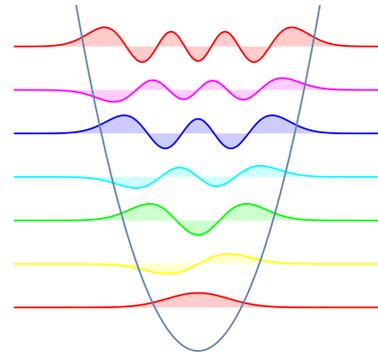


# LINEAR ALGEBRA

MATH 21B



## LINEAR TRANSFORMATIONS

**27.1.** If  $X, Y$  are linear spaces, we can look at **linear transformations**  $T$  from  $X$  to  $Y$ . This generalizes the case when  $X = \mathbb{R}^m, Y = \mathbb{R}^n$  and where the transformation  $T$  was given by a  $n \times m$  matrix  $A$ . We will in general no more have matrices to do that in general.

**27.2.**

**Definition:** A transformation  $T : X \rightarrow Y$  is called **linear**, if three conditions are satisfied. For all  $f, g \in X$  and  $k \in \mathbb{R}$  we have

- (i)  $T(0) = 0$
- (ii)  $T(f + g) = T(f) + T(g)$
- (iii)  $T(kf) = kT(f)$

**27.3.** The transformation  $D : C^\infty \rightarrow C^\infty$  which assigns to a function its derivative is one of the most important transformations for us. In the case  $X = Y$  one also talks about **operators**. The map  $D$  is an example of an operator. One calls it a **differential operator**.

**27.4.** If  $p$  is a polynomial, we can form  $p(D)$ . For example, for  $p(x) = x^2 + 3x - 2$  we obtain  $p(D) = D^2 + 3D + 2$  and get  $p(D)f = f'' + 3f' + 2f$ .

**Definition:**  $D(f) = f'$  and a polynomial  $p$ , define  $p(D)$ , a **differential operator**

**27.5.** The dimension of a linear space is the number of basis elements in it. The space  $C^\infty$  is infinite dimensional, but the kernel of differential operators can be finite dimensional.

**27.6.** For example, the dimension of the kernel of  $D$  is 1. The dimension of the kernel of  $D^3$  is 3. It consists of all polynomials which have degree less or equal than 3. We have seen in the lecture on homogeneous differential equations that:

**Theorem:** The dimension of the kernel of  $p(D)$  is equal to the degree of the polynomial  $p$ .

**27.7.** For example, when we solved the **harmonic oscillator differential equation**

$$D^2f + f = 0.$$

last week, we saw that the transformation  $T = D^2 + 1$  has a two-dimensional kernel. It is spanned by the functions  $f_1(x) = \cos(x)$  and  $f_2(x) = \sin(x)$ . Every solution to the differential equation is of the form  $c_1 \cos(x) + c_2 \sin(x)$ .

**27.8.** Let us look at the following linear transformation on  $2 \times 2$  matrices:  $T(A) = A^T$ . It maps

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

This is a linear transformation. We can see that the eigenvalues of  $T$  are 1 or  $-1$ . The eigenspace of the eigenvalue 1 contains the symmetric matrices, a space of dimension 3. The eigenspace to the eigenvalue  $-1$  are the anti-symmetric matrices, a space of dimension 1.

**27.9.** In quantum mechanics, the operator  $P = -i\hbar D$  is called the **momentum operator** and the operator  $Qf(x) = xf(x)$  is the **position operator**. The number  $\hbar$  is just a small constant. Every real  $\lambda$  is an eigenvalue of  $P$  with eigenfunction  $e^{i\hbar\lambda x}$ . This is called a **traveling wave**. One can now do cool computations like

$$[Q, P]f = QPf - PQf = -i\hbar x f'(x) + i\hbar \frac{d}{dx}(xf(x)) = i\hbar f(x)$$

We get the **canonical commutation relation**. Impress your friends.

**Theorem:**  $[Q, P] = i\hbar$

**27.10.** Here is an other important operator in quantum mechanics. The operator

$$Tf(x) = -f''(x) + x^2f(x)$$

is called the **energy operator** of the **quantum harmonic oscillator**. Don't worry about it too much. But you should be able to verify that it is linear.

We can check that  $f(x) = e^{-x^2/2}$  is an eigenfunction of  $T$  of the eigenvalue 1. Similarly, you can check that  $f(x) = xe^{-x^2/2}$  is an eigenfunction of  $T$  with the eigenvalue 3. One knows all the eigenfunctions of  $T$ . They are all of the form of a polynomial times  $e^{-x^2/2}$ . They are shown in the following picture:

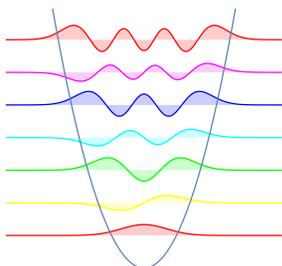


FIGURE 1. The eigenfunctions of the quantum harmonic oscillators describe standing waves.