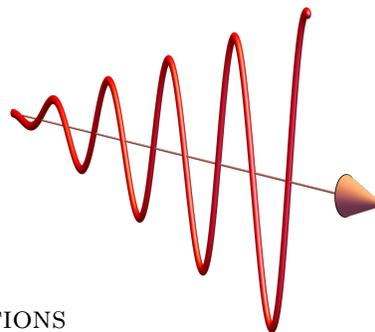


LINEAR ALGEBRA

MATH 21B



INHOMOGENEOUS DIFFERENTIAL EQUATIONS

28.1. If p is a polynomial we can look at the differential equations $p(D)f = g$, where g is a fixed function and f is the unknown. If $g = 0$, we have a **homogeneous differential equations**. We have seen them as the kernel of the **differential operator** $p(D)$. For example, we have seen that for $p(x) = x^2 + 1$, we got the differential equation

$$(D^2 + 1)f = 0$$

which means $f'' + f = 0$ which was solved by $f(t) = c_1 \sin(t) + c_2 \cos(t)$. The solution space is two-dimensional.

28.2. Earlier we have looked at systems of linear equations $A\vec{x} = \vec{b}$. We have seen that the solution space consisted of a particular solution $A\vec{x} = \vec{b}$ plus an element of the kernel of A . We now look at **differential equations** $p(D)f = g$, where g is an arbitrary smooth function. The situation is the same.

28.3.

Theorem: A differential equation $p(D)f = g$ has a n -parameter solution space, the sum of a particular solution and a solution of $p(D)f = 0$.

28.4. For example, if $p(x) = x^n$, we deal with the differential equation $D^n f = g$, meaning that we want to find a function f whose n 'th derivative is g . To get f , we have to integrate n times. Integration is the linear map $Sf(x) = \int_0^x f(t) dt$ giving a particular anti-derivative of a function. Since $DSf = f$, the function $f = S^n g$ solves $D^n f = g$. For example $f''' = \cos(x)$ is solved by $-\sin(x) + c_1 x^2 + c_2 x + c_1$.

28.5. This can be generalized. Instead of inverting D we just have to invert $D - \lambda$. We can solve $p(D) = a(D - \lambda_1)(D - \lambda_2) + \dots + (D - \lambda_n) = g$ we can invert each part. This is done by a **generalized integration formula**

Theorem: $(D - \lambda)^{-1}g = Ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} g(s) ds$.

So, in order to solve $(D - 1)(D + 2)f = g$ for example, we just use this “generalized integration” to get $f = (D - 1)^{-1}(D + 2)^{-1}g$. The **operation method** also verifies the theorem. Inverting each of the linear polynomials produces a new constant. The operation method is however not so easy to do. A machine likes it because there is nothing to think about.

28.6. The operator method always works. But it can produce a considerable amount of work. A cookbook method is to solve $p(D)x = g$ first by finding the **homogeneous solution** x_h which is the solution to $p(D)x = 0$. Then a **particular solution** x_p of the system $p(D)x = g$ is found by an educated guess. This method is faster and requires some "recipes". Fortunately, it is quite easy: as a rule of thumb: feed in the same class of functions which you see on the right hand side and if the right hand side should contain a function in the kernel of $p(D)$, try with a function multiplied by t . The general solution of the system $p(D)f = g$ is f is a sum of a homogeneous equation and a particular solution.

28.7. In the **second order case** $p(D) = (D - \lambda_1)(D - \lambda_2) = D^2 + bD + c$. The next table covers all cases for **homogeneous second order differential equations** $x'' + px' + q = 0$.

$\lambda_1 \neq \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$
$ik = \lambda_1 = -\lambda_2$ imaginary	$C_1 \cos(kt) + C_2 \sin(kt)$
$\lambda_1 = a + ik, \lambda_2 = a - ik$	$C_1 e^{at} \cos(kt) + C_2 e^{at} \sin(kt)$

28.8. Inhomogeneous solutions can be found in general by applying the operator inversions with $C = 0$ or by an educated guess. For $x'' = g(t)$ we integrate twice, otherwise, check with the following table:

$g(t) = a$ constant	$f(t) = A$ constant
$g(t) = at + b$	$f(t) = At + B$
$g(t) = at^2 + bt + c$	$f(t) = At^2 + Bt + C$
$g(t) = a \cos(bt)$	$f(t) = A \cos(bt) + B \sin(bt)$
$g(t) = a \sin(bt)$	$f(t) = A \cos(bt) + B \sin(bt)$
$g(t) = a \cos(bt)$ with $p(D)g = 0$	$f(t) = At \cos(bt) + Bt \sin(bt)$
$g(t) = a \sin(bt)$ with $p(D)g = 0$	$f(t) = At \cos(bt) + Bt \sin(bt)$
$g(t) = ae^{bt}$	$f(t) = Ae^{bt}$
$g(t) = ae^{bt}$ with $p(D)g = 0$	$f(t) = Ate^{bt}$
$g(t) = q(t)$ polynomial	$f(t) =$ polynomial of same degree

28.9. An important example is **resonance**. Lets look at $f'' + 9f = \sin(3t)$. The general homogeneous solution is $f(t) = c_1 \cos(3t) + c_2 \sin(3t)$. In order to find a particular solution, we would start with $f = A \sin(3t) + B \cos(3t)$. But if we plug this into the equation we get $0 = \sin(3t)$ which does not work. The cookbook tells you to **multiply with a t** and try $f = At \sin(3t) + Bt \cos(3t)$. Then $f'' + 9f = -6A \sin(3t) + 6B \cos(3t)$. We see that $A = -1/6$ and $B = 0$. The particular solution is $f(t) = -\sin(3t)/6$. The general solution is $\boxed{-t \sin(3t)/6 + c_1 \cos(3t) + c_2 \sin(3t)}$. See the Harvard demonstration team on youtube <https://www.youtube.com/watch?v=cPALfz-6pnQ>