

# LINEAR ALGEBRA

MATH 21B

## INNER PRODUCTS



**29.1.** The dot product allowed us to compute **distances** and **angles**. This can be carried over to general linear spaces. In general, we call this generalization an **inner product**.

**29.2.**

**Definition:** An **inner product**  $\langle f, g \rangle$  of two elements  $f, g$  in a linear space  $V$  is a real number attached to  $f, g$  so that conditions are satisfied:

- a)  $f \rightarrow \langle f, g \rangle$  is linear
- b)  $\langle f, g \rangle = \langle g, f \rangle$ .
- c)  $\langle f, f \rangle = 0$  implies  $f = 0$ .

**29.3.** In words, an inner product is a **bilinear positive-definite symmetric quadratic form**. Note that the symmetry and first condition implies that we have linearity in both  $f$  and  $g$ .

**29.4.** Given an inner product, there is a notion of **length**

$$\|f\| = \sqrt{\langle f, f \rangle}$$

and so **distance**  $\|f - g\|$  and a notion of **angle**

$$\cos(\alpha) = \frac{\langle f, g \rangle}{\|f\| \|g\|}.$$

This is defined because of the **Cauchy-Schwarz inequality** assures that the right hand side is in  $[-1, 1]$ .

**Theorem:**  $\langle f, g \rangle \leq \|f\| \|g\|$  for any inner product.

*Proof.* If  $g = 0$  the inequality is clear. If  $g$  is not zero, divide by  $\|g\|$  and assume  $\|g\| = 1$ . One has then to prove  $\langle f, g \rangle \leq \|f\|$ . As in Gram-Schmidt we look at  $f - \langle f, g \rangle g$ . Its length is  $\|f\|^2 + \langle f, g \rangle^2 \|g\|^2 - 2\langle f, g \rangle \langle f, g \rangle$  which is  $\|f\|^2 - \langle f, g \rangle^2$ . Because this is non-negative,  $\|f\|^2 \geq \langle f, g \rangle^2$  equivalent to the claim.  $\square$

**29.5.** 1) On  $\mathbb{R}^n$ , the standard **dot product**  $\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$  is an inner product.

2) On  $\mathbb{R}^{n,m}$  the product  $\langle A, B \rangle = \text{tr}(A^T B)$  is an inner product. You can check that this naturally generalizes the dot product because  $\langle A, B \rangle = A_{11}B_{11} + A_{12}B_{12} + \dots + A_{nn}B_{nn}$ .

3) On the space of continuous functions on  $[-\pi, \pi]$ , the inner product  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$  works. We will see this in Fourier theory. The product is chosen so that  $\|\cos(x)\| = \|\sin(x)\| = 1$ .

**29.6.** Lets look at some other cases.

4) Look on  $C^\infty$  the product  $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$ . The conditions a),b) are fine, but there are non-zero functions  $f$  for which  $\|f\| = 0$ . Indeed, any function which is zero on  $[0, 1]$  satisfies this condition.

5) Look on  $C[0, 1]$  at the product  $\langle f, g \rangle = f(0)g(0) + f(1)g(1)$ . Also this is not a valid inner product because  $\|f\| = 0$  for any function which is zero at 0 and 1.

**29.7.** Two elements are called **orthogonal** or **perpendicular** if  $\langle f, g \rangle = 0$ . This means that the angle between them is  $\pi/2 = 90^\circ$ . If we denote  $a = \|f\|, b = \|g\|, c = \|f - g\|$  as the lengths of a right angle triangle and  $\langle f, g \rangle = 0$ , we have the Pythagorean theorem

**Theorem:** The Pythagorean theorem  $a^2 + b^2 = c^2$  holds in any linear space with an inner product.

*Proof.*

$$\|f - g\|^2 = \langle f - g, f - g \rangle = \langle f, f \rangle + \langle g, g \rangle - 2\langle f, g \rangle = \|f\|^2 + \|g\|^2$$

□

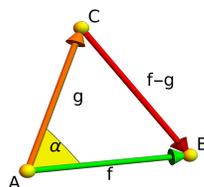


FIGURE 1. The Pythagorean theorem holds in any linear space with an inner product.

**29.8.** In statistics for example, where  $\text{Cov}[X, Y]$  is an inner product between centered random variables and  $\text{Cov}[X, X] = \text{Var}[X, X]$ , the Pythagorean theorem reads

$$\text{Var}[X] + \text{Var}[Y] = \text{Var}[X + Y],$$

where  $\text{Var}[X] = \langle X, X \rangle = \|X\|^2$ .



FIGURE 2. A compass. Once we have an inner product, we can do constructions. We can measure length. We can measure angles.