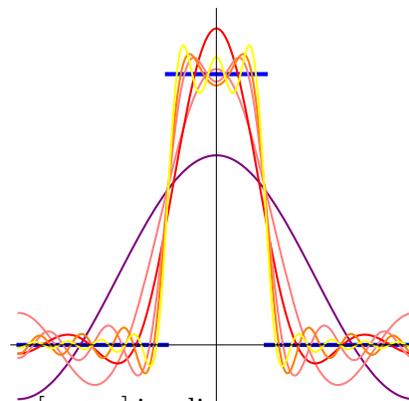


LINEAR ALGEBRA

MATH 21B

FOURIER ANALYSIS



30.1. The set $X = \mathbf{PS}[-\pi, \pi]$ of piecewise smooth functions on $[-\pi, \pi]$ is a linear space. Each of these functions also defines a periodic function. Let us look at the function $f(x) = x$. When continued as a function on \mathbb{R} , it produces a 2π periodic function. Such periodic functions appear for example when we listen to sound. Fourier theory allows for example to compress music. Our ear does Fourier theory when listening to speech or music. It is very useful.

30.2. If we take on X the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx .$$

we can define angles, length and projections in X , as we did in \mathbb{R}^n . We will especially project onto subspaces spanned by trigonometric polynomials.

30.3. Important is the fact that we have a **Fourier basis**:

Theorem: $\mathcal{B} = (\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots)$ is orthonormal.

30.4. To verify this, use the addition and double angle formulas in trigonometry.

30.5.

Definition: The numbers $a_0 = \langle \frac{1}{\sqrt{2}}, f \rangle$, $a_k = \langle f, \cos(kx) \rangle$ and $b_k = \langle f, \sin(kx) \rangle$ are called the **Fourier coefficients** of f .

30.6. Written out, this means $a_0 = \langle f, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx$, $a_k = \langle f, \cos(kx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$, $b_k = \langle f, \sin(kx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$.

30.7. The main goal of Fourier theory is to get to **Fourier series** or finite approximations of such a series.

Definition: The series $f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$ is called the **Fourier series** of f .

Theorem: $f(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$

30.8. At jump discontinuities of f , the Fourier series takes the average of the two boundary values. To make the formula precise, we would write the left hand side as $(f(x^+) + f(x^-))/2$. If we consider finite sums, this means we are looking at the projection of f onto the linear space of trigonometric polynomials of degree less or equal to n .

30.9. Remember that if V was a linear subspace of \mathbb{R}^n and $\vec{u}_1, \dots, \vec{u}_m$ is an **orthonormal basis** in V and \vec{x} is a vector, then $\text{proj}_V(\vec{x}) = c_1\vec{u}_1 + \dots + c_m\vec{u}_m$ was the **projection** of \vec{x} onto V .

30.10. The closest point on the on the $(2n + 1)$ -dimensional space of functions containing all trig polynomials degree of n or less is

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^n a_k \cos(kx) + \sum_{k=1}^n b_k \sin(kx) .$$

It is called the **n'th Fourier approximation of f** .

30.11.

Definition: A function is called **even** if $f(x) = f(-x)$.
A function is called **odd** if $f(x) = -f(-x)$.

30.12. If f is even and g is odd, then $\langle f, g \rangle = 0$. Therefore:

Theorem: if f is even then $f(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx)$ and if f is odd, then $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$.

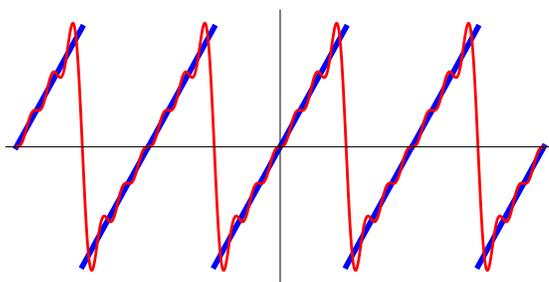


FIGURE 1. The 12'th Fourier approximation $g(x) = \sum_{k=1}^{12} b_k \sin(kx)$ of the function $f(x) = x$ on $[-\pi, \pi]$. We show the function periodically continued. The coefficients can be computed explicitly as $b_k = 2(-1)^{k+1}/k$.

30.13. For piecewise differentiable functions, we have convergence of the Fourier series to $(f(x^+) + f(x^-))/2$ at **every point**. This is **Dirichlet's theorem**. We will take this for granted. ¹

¹<https://people.math.harvard.edu/~knill/teaching/math22b2019/handouts/lecture30.pdf>