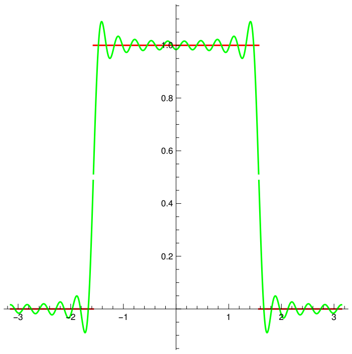

LINEAR ALGEBRA

MATH 21B

Fourier and Parseval

Joseph Fourier 1768-1830

32.1. Remember that a piecewise smooth function f on $[-\pi,\pi]$ has the Fourier coefficients $a_0 = \langle f, \frac{1}{\sqrt{2}}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}}\ dx$, $a_k = \langle f, \cos(kx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx$, $b_k = \langle f, \sin(kx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx$. The series $\frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$ is called the **Fourier series** of f. **Dirichlet's theorem on Fourier series** assures that at every point $x \in [0,2\pi]$, one has: [1]

Theorem: $f(x) = a_0\frac{1}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$

32.2. Example 1: We have seen $f(x) = x$ which was an odd function having therefore a *sin*-series. We have computed $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\ dx = 2(-1)^{n+1}/n$ and obtained

$$f(x) = \sum_{k=1}^{\infty} 2\frac{(-1)^{k+1}}{k}\sin(kx)\ .$$

Written out, this was

$$f(x) = \frac{2}{\pi}(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \dots)$$

32.3. Example 2: We then saw $f(x)$ which is 1 on $[-\pi/2,\pi/2]$ and 0 else. Because this has been an even function it has a cos series. We have computed $a_0 = 1/\sqrt{2}$ and $a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(nx)\ dx = \frac{2\sin(n\pi/2)}{n\pi}$ which is zero for even n and changes sign depending on n. This is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}(\cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - \dots)\ .$$

32.4. Because $\mathcal{B} = (\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots)$ is an orthonormal set, we have for every finite sum $f = \sum_{k=1}^{n} a_0\frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k\cos(kx) + \sum_{k=1}^{n} b_k\sin(kx)$

$$\|f\|^2 = a_0^2 + \sum_{k=1}^{n}\|a_k\|^2 + \sum_{k=1}^{n}\|b_k\|^2\ .$$

When we take the limit $n \to \infty$, we get the **Parseval's theorem**

Theorem: $\|f\|^2 = a_0^2 + \sum_{k=1}^{\infty}\|a_k\|^2 + \sum_{k=1}^{\infty}\|b_k\|^2$

[1] At discontinuities replace $f(x)$ with $(f(x-)+f(x+))/2$ and $f(0), f(2\pi)$ with $(f(0+)+f(2\pi-)/2$.

32.5. In **Example 1**, we have $\|f\| = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \; dx = \frac{\pi^2}{3}$. Parseval's theorem gives $\frac{\pi^2}{3} = 2(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots)$. This solves the **Basel problem**, first asked by Pietro Mengoli in 1650 and solved by Leonhard Euler

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6} \; .$$

32.6. In **Example 2**, we have $\|f\| = 1$. The coefficients satisfy $a_0^2 = \frac{1}{2}$ and $a_n^2 = \frac{\pi^2}{4n^2}$ if n is odd. Parseval's Theorem gives $1 = \frac{1}{2} + \frac{\pi^2}{4}\sum_{k \text{ odd}} \frac{1}{k^2}$. This means

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8} \; .$$

32.7. Example 3: The function $f(x) = 3\cos(x) + 2\sin(x)$ has norm $\|f\| = 5$. Can you see why Perceval's theorem generalizes the Pythagorean theorem?

32.8. Example 4: The function $f(x) = \frac{x^2}{2}$ has norm $\|f\|^2 = \frac{2}{\pi}\int_0^{\pi} (\frac{x^2}{2})^2 \; dx = \frac{\pi^4}{10}$ and the Fourier expansion

$$f(x) = \frac{\pi^2}{3\sqrt{2}}\frac{1}{\sqrt{2}} + \sum_{k=1}^{\infty} 2(-1)^k \frac{\cos(kx)}{k^2}$$

Parseval's theorem gives

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \; .$$

This number is $\zeta(4)$, the value of the **Riemann Zeta function** at $s = 4$.

32.9. Example 5: The function $g(x) = x^3 - \pi^2 x$ has the Fourier series

$$g(x) = \sum_{k=1}^{\infty} \frac{12(-1)^k}{k^3} \sin(kx) \; .$$

Perceval's theorem $\|g\|^2 = \sum_{k=1}^{\infty} b_k^2$ gives $\sum_k b_k^2 = \frac{2}{\pi}\int_0^{\pi} (x^3 - \pi^2 x)^2 \; dx = \frac{16\pi^6}{105}$ so that

$$\sum_k \frac{1}{k^6} = \frac{1}{144}\sum_k b_k^2 = \frac{1}{144}\frac{16\pi^6}{105} = \frac{\pi^6}{945} \; .$$

This number is the **Riemann Zeta function** at $s = 6$. Fourier allows to get explicit expressions of $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for every even integer s. For odd positive integers, $\zeta(s)$, there are no explicit expressions known.

32.10. Fourier developed the theory in the context of partial differential equations. It was historically important also because the concept of "function" got redefined. Before Fourier, functions were always thought of given by analytic expressions. A Heavyside function like in example 2 would not work. There is no Taylor series which approximates the function because all derivatives at 0 are zero. Fourier's claim of the convergence of the series was confirmed in the **19'th century** by Cauchy and Dirichlet. For continuous functions, the sum does not need to converge everywhere. It was only shown by **Fejér** in his PhD theses in 1900 that the Fourier coefficients still determine a continuous function function f on $[-\pi, \pi]$ provided that $f(-\pi) = f(\pi)$.