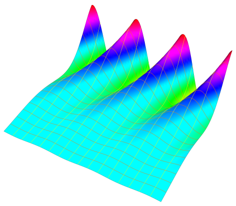
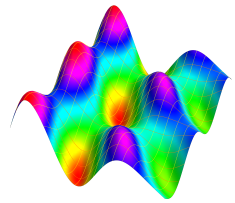


LINEAR ALGEBRA



MATH 21B

Partial Differential Equations

33.1. We look here at functions $f(t,x)$ which are for fixed t a piecewise smooth function in x. Analogously as we studied the motion of a **vector** $\vec{v}(t)$, we are now interested in the **motion of a function** f in time t. While the governing equation for a vector was an ordinary differential equation $x' = Ax$ (ODE), the describing equation is a **partial differential equation** (PDE) $f' = T(f)$.

33.2. The function $f(t,x)$ could denote the **temperature of a stick** at a position x at time t or the **displacement of a string** at the position x at time t. The motion of these dynamical systems will be easy to describe in the orthonormal Fourier basis like $\sin(nx)$. We have now to deal with derivatives with respect to two different variables

33.3. We write $f_x(t,x)$ and $f_t(t,x)$ for the **partial derivatives** with respect to x or t. The notation $f_{xx}(t,x)$ means that we differentiate twice with respect to x. Se still can use also the D notation for the derivative with respect to x. This would mean $Df(t,x) = f_x(t,x)$ and $f'(t,x) = f_t(t,x)$. Now, $f_t(t,x) = f_{xx}(t,x)$ would read as $f' = D^2 f$. If we could diagonalize D^2, then we could solve this partial differential equation. We will come to that.

Examples

Example: The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound. The function $f(t,x) = \sin(x-t) + \sin(x+t)$ satisfies the wave equation.

Example: The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic. The function $f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$ satisfies the heat equation.

Example: The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $f(x,y) = x^3 - 3xy^2$ is an example satisfying the Laplace equation.

Example: The **Poisson equation** $f_{xx} + f_{yy} = 12xy$ determines the shape of a membrane. The function $f(x,y) = x^3 y + xy^3$ is an example satisfying this Poisson equation.

Example: The **transport equation** $f_t = f_x$ is used to model transport in a wire. The function $f(t,x) = e^{-(x+t)^2}$ satisfies the advection equation.

Example: The **Burgers equation** $f_t + f f_x = f_{xx}$ describes waves at the beach which break. The function $f(t,x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}$ satisfies the Burgers equation.

Example: The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics. The function $f(x,y) = \cos(x) + \sin(y)$ satisfies the eiconal equation.

Example: The **KdV equation** $f_t + 6 f f_x + f_{xxx} = 0$ models **water waves** in a narrow channel. The function $f(t,x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2 t))$ satisfies the KdV equation.

Example: The **Schrödinger equation** $f_t = \frac{i\hbar}{2m} f_{xx}$ is used to describe a **quantum particle** of mass m. The function $f(t,x) = e^{i(kx - \frac{\hbar}{2m} k^2 t)}$ solves the Schrödinger equation. [Here $i^2 = -1$ is the imaginary i and $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.]

Example: The **Black-Scholes equation** $f_t = f - x f_x - x^2 f_{xx}$ is used in finance. Here $f(t,x)$ is the prize of a **call option** and x the stock prize and t is time. Example solutions are $f(t,x) = x$ or $f(t,x) = e^t$.

Paul Dirac once said: "A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an **antiparticle**: the existence of antiparticles was later confirmed.

33.4. We will learn next soon how to use Fourier theory to solve partial differential equations. For now, you should be able to check that a simple function satisfies a partial differential equations and already know the heat and wave equation.

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