LINEAR ALGEBRA
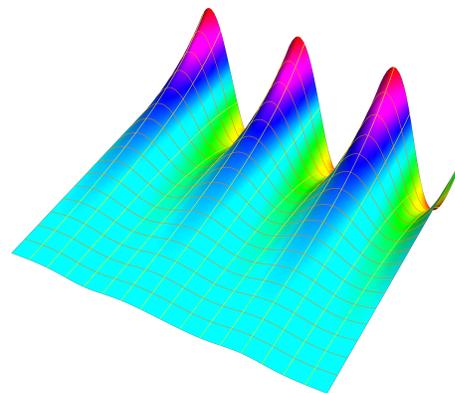
THE HEAT EQUATION

34.1. The partial differential equation

$$f_t = \mu f_{xx}$$

is called the **heat equation**. It is an equation for an unknown function $f(t,x)$ of two variables t and x. The constant μ is a parameter. The interpretation is that $f(t,x)$ is the temperature at **time** t and **position** x. In order to use Fourier theory, we assume that f is a function on the interval $[-\pi, \pi]$. The problem is: given an initial heat distribution $f(0,x)$, what is the situation $f(t,x)$ at a later time? The process does what one expects from heat. It produces **diffusion**.

34.2. What does the equation tell? We have a temperature distribution $x \to f(t,x)$. The rate of change in time of this temperature is the second space derivative of f. If x is a location, where $f(t,x)$ is concave down as a function of x, this means that f_t is negative and that the function will decrease there in the near future. If $f(t,x)$ is concave up, then this means that f_t is positive, meaning that f increases there. While the partial differential equation describes a motion of a function f the set-up is as before, where we looked at the motion $v(t)$ of a vector v.

34.3. In order to use the **closed-form solution method** from the earlier part of the course, we write the heat equation as

$$f_t = \mu D^2 f$$

and think of $A = \mu D^2$ as a transformation or a matrix. Now remember what we did in the case of differential equations $x' = Ax$; we found the eigenvalues and eigenvectors of A. This is what we do here too. But we know already that $\cos(nx), \sin(nx)$ are eigenfunctions of D^2 to the eigenvalue $-\mu n^2$. In the ordinary differential equation case, we also expressed $v(0) = c_1 v_1 + \cdots + c_n v_n$ as a sum of eigenfunctions, then wrote down the **closed-form solution** $v(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n$.

34.4. We do the same thing here and can use that the Fourier basis is an eigenbasis of D^2. This is the great discovery of Fourier:

> **Theorem:** $f(t,x) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t} \sin(nx)$ solves $f_t = f_{xx}$.

Proof. For $t = 0$, we get the Fourier series of $f(0,x)$. A direct differentiation shows that $f_t = \mu f_{xx}$. ☐

34.5. Fourier already used the sin series. We follow his steps and continue the function as an odd function on $[-\pi, \pi]$ and use only the sin-series. We could also continue the function as an even function on $[-\pi, \pi]$ and use the cos-series. The later is done in applications like JPG encoding of pictures as there, the average is important as it represents brightness. Let us look at an example of $f(0, x) = \sin(x)$ on $[0, \pi]$. If we continue that as an odd function, then this is just $\sin(x)$ and $f(t, x) = e^{-t^2} \sin(x)$ solves that equation. If we continue the function as an even function, then we deal with $f(x) = |\sin(x)|$.

34.6. Problem: Find the general solution of the modified heat equation $f_t = 3f_{xx} + f$, where $f(0)$ is 1 for $x \in [\pi/3, 2\pi/3]$ and 0 else. Let us assume that f stays zero at the boundary of $[0, \pi]$ and is continued in an odd way so that it has a sin-series. This equation is a driven heat equation which can model a **fire** which in which heat produces more fuel. **Solution:** f has a sin series with

$$b_n = \frac{2}{\pi} \int_{\pi/3}^{2\pi/3} \sin(nx) \, dx = \frac{2}{n\pi}[-\cos(2n\pi/3) + \cos(n\pi/3)] .$$

Now look the operator $A = 3D^2 + 1$ on the right hand side so that $f_t = Af$. What are the eigenvalues of A? Since the eigenvalues of D^2 are $-n^2$, the operator A has the eigenvalues $\lambda_n = -3n^2 + 1$. The closed-form solution is $f(t, x) = \sum_{n=1}^{\infty} b_n e^{(-3n^2+1)t} \sin(nx)$.

34.7. Problem: Let us take the same problem as before but increase the fuel strength feeding the fire $f_t = 3f_{xx} + 5f$. What happens now is that

$$f(t, x) = \sum_{n=1}^{\infty} b_n e^{(-3n^2+5)t} \sin(nx) .$$

The high frequency parts still die out but there is one mode which explodes now exponentially because $\lambda_n = -3n^2 + 5$. The fire takes over. Let us write down the solution in a bit more intelligible way. The coefficient $b_n = -\cos(2n\pi/3) + \cos(n\pi/3)$ is 0 for even n, It is alternating 1 and -2 for odd n. We have the initial condition

$$f(0, x) = \frac{1}{\pi}\left(\frac{\sin(x)}{1} - \frac{2\sin(3x)}{3} + \frac{\sin(5x)}{5} - \cdots\right) .$$

The solution of the heat equation is now

$$f(t, x) = \frac{1}{\pi}\left(\frac{e^{2t}\sin(x)}{1} - \frac{e^{-7t}2\sin(3x)}{3} + \frac{e^{-22t}\sin(5x)}{5} - \cdots\right) .$$

It is the e^{2t} part which renders the fire out of control.

34.8. Problem: Let us look at $f_t = 3f_{xx} + t^7$ with initial condition $f(0, x) = \sin(4t)$. First look at the solution of $f_t = 3f_{xx}$ and call this the **homogeneous solution**. The solution is $e^{-9t} \sin(4t)$. Now find a solution $f(t)$ which only depends on time. The differential equation $f'(t) = t^7$ is solved by $t^8/3$. The sum of the homogeneous and particular solution is

$$f(t, x) = e^{-3*4^2 t} \sin(4x) + \frac{t^8}{8} .$$