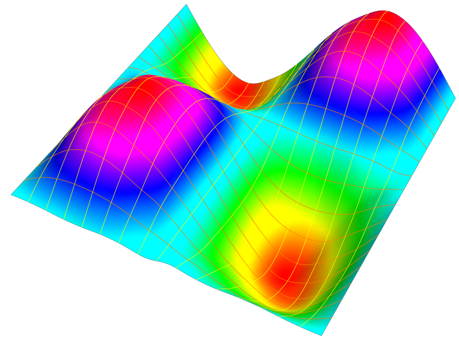
LINEAR ALGEBRA

MATH 21B



THE WAVE EQUATION

LECTURE

35.1. The partial differential equation

$$f_{tt} = c^2 f_{xx}$$

is called the **wave equation**. It is an equation for an unknown function $f(t,x)$ of two variables t and x. The constant c is a parameter. The interpretation is that $f(t,x)$ is the string amplitude at **time** t and position x. The constant c is the speed of the wave. Again, we assume that f is a function on the interval $[-\pi,\pi]$. One problem is: given an initial string position $f(0,x)$, what is $f(t,x)$ at a later time? Another problem is to give the initial velocity $f_t(0,x)$ and determine from this the string position at time t. We can also give both the initial position and velocity in which case we just add up the solution for the initial position and the solution for the initial velocity.

35.2. What is the meaning of the wave equation? We can interpret the acceleration f_{tt} as a **force** acting on the string. By Newton's law, acceleration is proportional to force. That force acts in such a way that the string wants to be flattened out. But as the system not only has position but also momentum, it does not just flatten out as in the heat case. It is a **conservation of energy** which prevents the system to settle. Indeed, the **energy functional** $H(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_t^2 + c^2 f_x^2 \, dx$ is constant. [1]

35.3. For the heat equation, the solutions $\boxed{x(t) = e^{\lambda t} x(0)}$ of $\boxed{x'(t) = \lambda x}$ was important. For the wave equation, the solution $x(t) = x(0)\cos(ckt) + x'(0)\frac{\sin(ckt)}{c}$ of $\boxed{x''(t) = -c^2k^2x}$ is important. We see that if $\sin(kx)$ is the initial position, then $f(t,x) = \cos(ckt)\sin(kx)$. By adding this up, we get:

Theorem: If $f_t(0,x) = 0$ and $f(0,x) = \sum_{k=1}^{\infty} b_k \sin(kx)$, the solution is $f(t,x) = \sum_k b_k \cos(ckt)\sin(kx)$.

To see this, just check that $f_t(0,x) = 0$ and that $f(0,x)$ agrees with the Fourier expansion of $f(0,x)$.

[1] $\frac{d}{dt} 2\pi H(f) = \int_{-\pi}^{\pi} 2 f_t f_{tt} + 2c^2 f_x f_{tx} \, dx = \int_{-\pi}^{\pi} 2c^2 f_t f_{xx} + 2c^2 f_x f_{tx} \, dx = \int_{-\pi}^{\pi} \frac{d}{dx}(2c^2 f_t f_x) \, dx = 0.$

35.4. The solution to the **harmonic oscillator** $x''(t) = -c^2k^2x$ also has a contribution $x'(0)\sin(ckt)/(kc)$ which takes care of the **initial velocity**. This allows us to write down the **closed-form solution** if the initial velocity is given.

Theorem: If $f(0,x) = 0$ and $f_t(0,x) = \sum_k b_k \sin(kx)$, the solution is $f(t,x) = \sum_{k=1}^{\infty} \frac{b_k}{ck} \sin(ckt)\sin(kx)$.

To verify this, just look what happens for $t = 0$. We have $f(0,x) = 0$ and $f_t(0,x)$ agrees with the Fourier expansion of $f_t(0,x)$.

35.5. Example 1: What is the solution of the driven wave equation

$$f_{tt} = 4f_{xx} + 6t$$

if $f(0,x) = \sum_k \frac{1}{k^3}\sin(kx)$. We first solve the **homogeneous problem**

$$f_{tt} = 4f_{xx} \ .$$

The solution is

$$f(t,x) = \sum_k \frac{\cos(2kt)}{k^3}\sin(kx) \ .$$

A special solution which does not depend on x satisfies $f_{tt} = 6t$ which has the solution t^3. The final solution is

$$f(t,x) = t^3 + \sum_{k=1}^{\infty} \frac{\cos(2kt)}{k^3}\sin(kx) \ .$$

35.6. Example 2: A piano string is fixed at the ends $x = 0$ and $x = \pi$ and is initially undisturbed $f(0,x) = 0$. The piano hammer induces an initial velocity $f_t(x,0) = g(x)$ onto the string, where $g(x) = \sin(3x)$ on the interval $[-\pi/2, \pi/2]$ and $g(x) = 0$ on $[\pi/2, \pi]$ or $[-\pi, -\pi/2]$. How does the string amplitude $f(t,x)$ move, if it follows the wave equation $f_{tt} = f_{xx}$?

Solution: The Fourier series has the coefficients $b_k = (2/\pi)\int_0^{\pi/2} \sin(3x)\sin(kx)\,dx = (2k\cos(k\pi/2))/(\pi(k^2-9))$. The solution is

$$f(t,x) = \sum_k b_k \frac{\sin(kt)}{k}\sin(kx) \ .$$

35.7. Example 3: Solve the partial differential equation $f_{tt} = 16f_{xx}$ with initial condition $f_t(0,x) = x^3$ and $f(0,x) = x^3$.

Solution: The initial condition is $b_k = (2/\pi)\int_0^{\pi} x^3\sin(kx)\,dx$ which can be computed using integration by parts. The answer is

$$b_k = 2(6 - k^2\pi^2)(-1)^k/k^3 \ .$$

The solution is

$$f(t,x) = \sum_{k=1}^{\infty} b_k\cos(4kt)\sin(kx) + \sum_{k=1}^{\infty} b_k\frac{\sin(4kt)}{4k}\sin(kx) \ .$$