The Nine Chapters on the Mathematical Procedures and Liu Hui's Mathematical Theory

Guo Shuchun

Abstract When discussing ancient mathematical theories, scholars often limit themselves to Greek mathematics and, especially to its axiomatic system, which they use as the standard to evaluate traditional mathematics in other cultures: whichever failed to form an axiomatic system is considered to be without theory. Therefore, even those scholars who highly praise the achievements in ancient Chinese mathematics consider that "the greatest deficiency in old Chinese mathematical thought was the absence of the idea of rigorous proofs" and that there is no formal logic in ancient Chinese mathematics; in particular it did not have deductive logic. They further contend that, "in the flight from practice into the realm of pure intellect, Chinese mathematics did not participate," [5, p. 151][1] and conclude that Chinese mathematics has no theory.

I think that Liu Hui's commentary (263 A.D.) to the Nine Chapters on the Mathematical Procedures, hereafter Nine Chapters, completely proved the formulas and solutions in Nine Chapters. It, mainly based on deductive logics, elucidated deep mathematical theories. Even though Nine Chapters itself does not contain mathematical reasoning and proofs, which is a major flaw in the pursuit of mathematical theory in the history of Chinese mathematics, there are certain correct abstract procedures that possess a general applicability which should be considered as mathematical theories in Chinese mathematics. Sir Geoffrey Lloyd, after explaining the difference between Liu Hui's and Euclid's mathematics, said "Mais cela ne signifie pas une absence d'intérêt pour la validation des résultats ou pour la recherche d'une systématisation" [3, préface, p. xi]. Based on the Nine Chapters, this article will discuss Liu Hui's contribution to the mathematical theory in order to stimulate more fruitful discussions.

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[1] Needham cited the claim that Chinese mathematics lacks "rigorous proofs," from his correspondence with the Japanese historian of Chinese mathematics, Yoshio Mikami [三上義夫].

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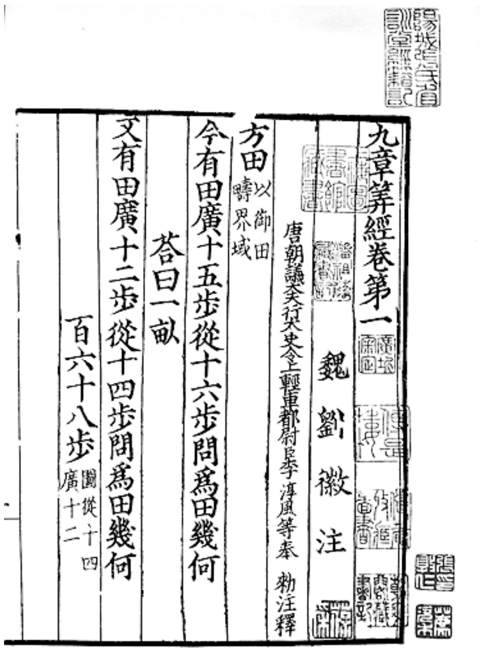
1 Procedures in Nine Chapters and its Style in which Questions Associated with Procedures as Examples

Many describe Nine Chapters on the Mathematical Procedures [九章算術 *Jiuzhang suanshu*], as a collection of application problems. Generally speaking, such a simplified view is not too off target; however, it needs clarification and merits more discussions if such a view leads to the misunderstanding that the ancient Chinese mathematics had no theories. The simple truth is that many who have not studied Nine Chapters or who had but did not seek to understand fully presume that, based on this simplified view, Nine Chapters consists of collections of one question, one answer, and one computational procedure [術 *shu*]. They further assume that these procedures are in effect the concrete solutions to the application problems. Such assumption is completely off base. It does not provide proper and accurate descriptions of Nine Chapters at all.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荅曰一畝
又有田廣十二步從十四步問爲田幾何
百六十八步 圖從十四 廣十二

Fig. 1 The first page of Chapter One in the Nine Chapters, the edition from Southern Song Dynasty (1127–1279)

In fact, the procedures in Nine Chapters are of different levels of abstractness and applicability. The relations among problems, answers, and procedures, or, to put it differently, the styles in the treatise are rather complicated. We first consider the relations among the problems, answers, and procedures. Roughly speaking, there are two different genres:

1. Questions associated with procedures as examples:

For this genre, there are usually multiple procedures with multiple questions, or one procedure with multiple questions, or one procedure with one question. But they can be further divided into three types of different scenarios:

(1) The text first listed one or multiple questions and then provided one or several general abstract procedures; moreover, the questions are only listed the statement and the answers without calculating procedures containing specific numeric values from the questions.[2]

Take the procedure of finding the area of the circular field in the Rectangular Field [方田 *Fangtian*] Chapter:[3]

> Now there is a circular field, the circumference of which is 30 *bu*[4] and the diameter of which is 10 *bu*. Question: what is [the area of] the field?
> Answer: 75 [square] *bu*.
> And there is another circular field, the circumference of which is 181 *bu* and the diameter of which is 60 1/3 *bu*. Question: what is [the area of] the field?
> Answer: 11 *mu*[5] 90 1/12 [square] *bu*.
> Procedure: The half-circumference multiplied by the half-diameter will yield the area in [square] *bu*.

The procedure is equivalent to the formula for the area of a circle:

$$S = \frac{1}{2}Lr, \tag{1}$$

where S, L, and r are the area, the circumference, and the radius of the circle. Here these two questions only have the statement of the questions and answers without

[2] By an "abstract" procedure, I mean a procedure containing the general description of measurements needed without any numeric values. In particular, it does not utilize the specifically numerical values appearing in the statement of a question. Instead, the procedure prescribes one or a series of operations to be performed on measurements without numerical values in words. The general description of measurements for example can be the diameter or the circumference of a circle or the number of days for a wild goose to fly from the south sea to the north sea. In this view, procedures of this kind in Nine Chapters are similar to mathematical formulas described with letters *a*, *b*, and *c* in the modern form.

[3] I use two references [6] and [3] for Nine Chapters and Liu Hui's [劉徽] commentary. For the circular field method [圓田術 *yuantianshu*], see [6, pp. 18–19] and [3, pp. 176–179]. Below pages from both references will be given, for all the text and examples from the Nine Chapters.

[4] *bu* [步] is a unit of length, 1 *bu* is equal to 6 *chi*. The areas and the volumes in the Chinese texts were, however, also expressed in terms of *bu*. The context made it clear whether the *bu* represents the linear *bu*, the area bu^2, or the volume bu^3. The units discussed below, *zhang*, *chi*, and *cun* all share this characteristic of representing the linear length, the area, and the volume.

[5] *mu* [畝] is a unit of area, equal to 240 *bu*.

individual procedures; the circular field method is the procedure for both questions. Some studies describe this procedure as the procedure for the 32nd question. This description is obviously not accurate. In Nine Chapters, the entire Rectangular Field Chapter, the jinglü [経率 *jinglü*] procedure, qilü [其率 *qilü*] procedure, and inverse qilü [反其率 *fanqilü*] procedure in the Millet and Rice [粟米 *Sumi*] Chapter, four root-extracting [開方 *kaifang*] procedures in the Small Width [少広 *Shaoguang*] Chapter, many procedures in the Work Discussing [商功 *Shanggong*] Chapter, the four fair labor [均輸 *junshu*] procedures in the Fair Labor Chapter, five excess-deficiency [盈不足 *yingbuzu*] procedures in the Excess-Deficiency Chapter, and five procedure in the Right Triangle [勾股 *Gougu*] Chapter are of the type. In the Nine Chapters, there are a total of seventy three procedures and one hundred and six questions of this type.

(2) The text first provided abstract procedures and then listed a few examples, which have the statements of questions and answers without procedures: Take flat-headed stack [芻童 *Chutong*] procedure for finding the volume of a frustum in the Work Discussing Chapter as an example.[6]

> Procedure: Double the upper length and add it to the lower length; also double the lower length and add it to the upper length; multiply them with their corresponding width. Add the products together. Use the height or the depth to multiply the sum and divide by 6.
>
> Now there is a frustum with the lower length equal to 2 *zhang*[7], upper length 3 *zhang*, the lower width is 3 *zhang*, the upper width is 4 *zhang*, and the height is 10 *zhang*. Question: What is the volume?
> Answer: 26500 [cubic] *chi*.[8]

The procedures and questions of finding volumes of other solids are exactly like this. The questions following the procedure also only have statement of questions and answers without individual procedures. The procedure preceding the questions is their common procedure. In this type, there are two general procedures with 10 questions.

(3) The text first provided general procedures and then listed a few questions, each of which contained the statement of question, the answer, and the calculating procedure with specific numeric values from the question. The procedure for each question is basically the application of the general procedure on that particular question. To demonstrate this type, we use the 'suppose' [今有 *Jinyou*] procedure in the Millet and Rice Chapter and some of the 31 questions of conversion involving grains and rice ([6, pp. 70–78] and [3, pp. 222–227].):

> 'Suppose' procedure: use the product of the given quantity [of grain in possession] and the ratio [of the] desired [grain] as the dividend, and the ratio [of the possessed grain] as the divisor.

[6] Here a flat-headed stack is a solid obtained by cutting a rectangular pyramid with a plane parallel to the base of the pyramid and removing the top part. The flat-headed stack procedure and the example can be found in [6, pp. 185–186] and [3, pp. 434–439].

[7] *zhang* [丈] is a unit of length, equal to 10 *chi*.

[8] *chi* [尺] is a unit of length, which is about 23–24 cm.

> Divide the dividend with the divisor. Now there is one *dou*[9] of millet and [we] want to exchange it for coarse rice [糲米 *limi*]. Ask: how much [of coarse rice] can be gotten?
> Answer: [We] get six *sheng*[10] of coarse rice.
> Procedure: Use the millet to exchange for the coarse rice. Multiply [the quantity of millet] by 3 and then divide [the product] by 5.
> Now there are 2 *dou* and one *sheng* of millet and [we] want to exchange it for polished rice [稗米 *baimi*]. Ask: how much [of polished rice] can be gotten?
> Answer: One *dou*, one and 17/50 *sheng* of polished rice.
> Procedure: Use millet to exchange for polished rice. Multiply [the quantity of millet] by 27 and then divide [the product] by 50.

Following these questions and procedures there are 29 questions of the same type. Each question is followed by its answer and a procedure; and each procedure is an application of the 'suppose' procedure, and therefore we do not repeat them here. For this type, each problem has the statement of the question, the answer, and its own procedure, which is an application of the 'suppose' procedure. Also belonged to this type are the proportional distribution [衰分 *cuifen*] procedure and inverse proportional [返衰分 *fangcuifen*] procedure with their nine examples in the Proportional Distribution Chapter, the small width procedure with its eleven examples in the Work Discussing Chapter, the eleven examples solved by excess-deficiency procedure in the Excess-Deficiency Chapter, and the rectangular array [方程 *fangcheng*] procedure, sign [正負 *zhengfu*] procedure, and loss and gain [損益 *sunyi*] procedure with their eighteen examples in the Rectangular Array Chapter. In total, there are 7 general procedures, 80 questions, and 78 sub-procedures with numerical values from the individual questions.

The above three types has a total of 82 general procedures, 196 questions, sub-procedures with numerical values, constituting 80% of Nine Chapters.

2. The collections of application problems:

In this genre, the text usually consists of one question, one procedure, and one answer. The degree of generality of the procedures varies:

(1) General procedures applicable to one kind of questions. Take the question of wild duck and wild goose in the Fair Labor Chapter as an example [6, p. 254] and [3, pp. 532–533]:

> Now a wild duck coming from the South Sea takes 7 days to reach the North Sea; a wild goose leaving from the North Sea takes 9 days to reach the South Sea. Now both birds leave [from their respective place at the same time]. Ask: when will they meet?
> Answer: 3 and 15/16 days.
> Procedure: Take the sum of the numbers of days as the divisor and the product of the days as the dividend. Divide the dividend by the divisor to get the answer.

This procedure, although did not depart from the subject of days in the question, did not include the actual numerical values from the question. It is applicable to

[9] *dou* [斗] is a unit of capacity, equal to 10 *sheng*.

[10] *sheng* [升] is a unit of capacity, which is about 198–210 ml.

many questions of the same nature. Many problems in the Fair Labor Chapter and the question of taking a pole and walking out of the door in the Right Triangle Chapter all belong to this type [6, p. 422] and [3, pp. 742–743].

(2) The actual computation for the concrete questions. Take the question of a door ajar with certain distance from the door threshold in the Right Triangle Chapter as an example [6, p. 413] and [3, pp. 714–717].:

> Now there is a door ajar and [the ends of the two opened panels] are 1 *chi* from the door threshold and 2 *cun*[11] apart. Ask: what is the width of the door?
> Procedure: Multiply the one *chi* by itself. Divide the result by one half of the distance between the two ends, two *cun*. Then add to the result by one-half of the distance between the two ends to get the width of the door.

The procedure incorporates the concrete numbers from the question; therefore it cannot exist independently from the question. The problems of inverse proportional distribution [非衰分 *feicuifeng*] in the Proportional Distribution Chapter, some examples in the Fair Labor Chapter, the Jade and Stone Hide Each Other [玉石互隱 *yushi huyin*] question in the Excess-Deficiency Chapter, and the solutions to right triangle questions and the measuring-the-height-of mountain-from-a-tree question in the Right Triangle Chapter are of the same type. These two types can be described as "a collection of application problems." There are 50 questions of this genre in Nine Chapters.

Obvious from the analysis above is that, it is not appropriate to summarize Nine Chapters as a collection of application problems. It is far from the truth that the text follows the "one-question-one-answer-and-one-procedure" format. In my opinion, there are at least three different formats for mathematical treatises in the history of mathematics. The first is represented by Euclid's Elements, forming an axiomatic system. The second format is collections of application problems, e.g. Diophantine' Arithmetics; to paraphrase the words of the German historian, Henkel, "one is still puzzled by the 101st question after carefully studying the first 100 questions." Obviously for the most part in Nine Chapters, the text does not conform to either of the formats in Euclid's Elements or in Diophantine's Arithmetics; therefore its format should be considered in its own right as the third type, i.e. the format centered around procedures with the questions associated with procedures as examples.

Meanwhile, it is not hard to see from the above analysis that the procedures in Nine Chapters are not of a single nature in terms of their abstractness and general applicability. At the very least, there are three varieties. The procedures of the first kind, in spite of the variations of expressions, share several common characteristics: the statement of the procedure is the core. Associated with the procedures are questions as demonstrations of the procedures. The procedures as the core are very abstract, rigorous, and with wide general applicability. When converted to the modern notion, the procedures become mathematical formulas or the operation procedures. These procedures are constructive and mechanical. Therefore, we describe the format of these procedures as "questions associated with procedures as examples." The

[11] *cun* [寸] is a unit of length, equal to one-tenth of a *chi*.

second kind of procedures can be described as abstract procedures connected to one type of questions with relatively wide applicability. The third kind consists of the actual computations of questions with actual numerical values.

The former two varieties constitute 90% of the text in Nine Chapters. These abstract procedures with general and relatively wide applicability should certainly be recognized as an expression of the mathematical theory.

2 Liu Hui's Mathematical Definitions and Deductive Reasoning

In order to describe Liu Hui's contributions to the mathematical theory, we need to examine Liu Hui's mathematical definitions and reasoning, especially his deductive reasoning.

2.1 Liu Hui's Definitions

Following the tradition of providing definitions to concepts in the Classic of Mo [墨経 *Mojing*], Liu Hui provided many rigorous definitions to many mathematical concepts. For example, the definition of power [冪 *mi*] as area:

> Whenever the width is multiplied with the length, [the result is what we] call *mi*

Another example is about the definition of *fangcheng* [方程] as the system of linear equations [6, p. 353] and [3, pp. 616–617]. *Fang* [方] means juxtaposing. For *cheng* [程] Liu Hui's annotation says:

> '*Cheng*' means to find the standard [of objects]. A group of objects is mixed together. Each row has [unknown] numbers of objects. The total sum of the products of the numbers and the objects is expressed. Set the *lü* [率] for each column.[12] If there are two unknown quantities, make the second column; If there are three unknown quantities, make three columns. Make the number of columns according to the number of unknowns. List one column after another; that is what we call *Fangcheng*. The rows do not depend on those besides them and each of them is based on the information given. [程, 課程也. 群物総雜. 各列有数. 総言其實. 令每行為率. 二物者再程. 三物者三程. 皆如物数程之. 竝列為行. 故謂之方程. 行之左右無所同存. 且為有所據而言耳.]

These are definitions of operations; that is, the definition can be carried out to obtain what is being defined.

It is worth pointing out that once the definition of a term was given, generally speaking, the term maintained the same connotation throughout the entire Nine Chapters.

[12] Each object might have two numbers associated with it, one known and the other unknown. For example, the number of chickens is known but the price of them isn't. But the sum of the unknown quantities is given as *shi*. The concept of *lü* in the Nine Chapters is rather complicated. I refer the readers to the discussion in [3, pp. 956–959].

2.2 Liu Hui's Deductive Reasoning

Many scholars believe that the ancient Chinese mathematics never used formal logic. This is fundamentally false. Not only did Liu Hui employ "Learning by analogy [举一反三 *juyi fansan*] Learning by consequence [告往知来 *gaowang zhilai*], and draw an analogy [触類而長 *chulei erzhang*]"[13] analogous method to expand mathematical knowledge, he also utilized formal logic in his general reasoning. He not only used inductive but also deductive reasoning.
Examples:

(1) Syllogism

There are abundant examples in which Liu Hui employed syllogism. For example, in the excess-deficiency procedure, Liu Hui described the situation when both guessed answers were assumed to be fractions ([6, p. 308] and [3, pp. 560–561]):

> Commentary: if both guessed answers are fractions, find the common denominator and arrange the numerators accordingly. For this question, both guessed answers are fractions, and therefore find the common denominator and arrange the numerators accordingly.

Take M to be "both guessed answers are fractions;" P, "one should find the common denominator and arrange the numerators accordingly;" and S, as this particular problem to be solved. This reasoning has and only has three concepts: (1) the two assumptions are fractions (the middle item, M), (2) find the common denominator and arrange the numerators accordingly (former presupposition P), and (3) this problem (latter presupposition S). Therefore Liu Hui's reasoning fits perfectly the AAA rule of syllogism:

The former inference:	M $\rightarrow$ P	(A)
The latter inference:	S $\rightarrow$ M	(A)
The conclusion:	S $\rightarrow$ P	(A)

(2) Relational Reasoning

As a commentator of a mathematical treatise, Liu Hui used many relational reasoning, which are special cases of syllogism. Among the relation judgments made by Liu Hui, the majority of them are of the equality relation. For example, right after Liu Hui proved one formula for the area of a circle, Liu Hui stated the following in order to prove another formula $S = \frac{1}{4}Ld$, [6, p. 23] and [3, pp. 560–561]:

> Multiply one-half of the circumference and one-half of the diameter [to get the area of a circle]. But now both the circumference and the diameter are full, therefore the two [numbers in the] denominator should be multiplied [to get] four. [That is,] use 4 to divide the quantity.

That is to say,

[13] These terms are used to describe the teaching method in which students when shown one example are expected to understand the analogous problems. These term were Liu Hui's words in the preface of Nine Chapters [6, p. 2] and [3, pp. 128–129].

the known formula: $S = \frac{1}{2}Lr$ (equality relation) and

$r = \frac{1}{2}d$ (equality relation), therefore

$S = \frac{1}{2}L \times \frac{1}{2}d = \frac{1}{4}Ld.$ (equality relation)

The equality relation is reflective and transitive.

Moreover, Liu Hui also used inequality relation in his reasoning. The operation of sphere volume [開立圓 *kai liyuan*] procedure in Nine Chapters utilized an incorrect formula for the volume of a sphere:

$$V = \frac{9}{16}d^3,$$

where V is the volume of a sphere and d is the diameter.[14] Liu Hui recorded how this incorrect formula was derived: Take a cube with the length of its side equal to the diameter of the sphere d. The ratio of volumes of the cube and the transcribed circular cylinder inside it is 4:3; the ratio of the volumes of a circular cylinder and the transcribed sphere inside it is also 4:3, taking 3 as the value of the circumference-diameter ratio. Therefore the ratio of the volumes of the cube and its transcribed sphere should be 16:9. That was how the above formula was derived according to Liu. Liu Hui took the intersection of two perpendicular circular cylinders of the same diameter and called it matching square cover [牟合方蓋 *mohe fanggai*] in Figure 2. Then he described [6, p. 142] and [3, pp. 378–381]:

> The square cover has a ratio related to squares; a transcribed sphere in it, a ratio related to circles. Deducing from this fact, [we ask,] how can those who described the circular cylinder as related to the ratio related to squares be not wrong?

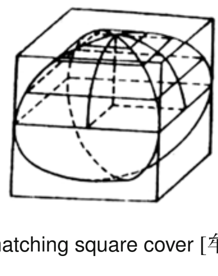


Fig. 2 matching square cover [牟合方蓋]

His lines of reasoning go as follows:

[14] This procedure describes how to find the diameter of a sphere when its volume is given. The procedure and Liu's commentary can be found in [6, pp. 141–143] and [3, pp. 378–385].

$$\text{matching square cover : a sphere} = 4 : \pi$$
$$\text{A circular cylinder : a sphere} \neq \text{matching square cover : a sphere}$$
$$\text{Hence a circular cylinder : a sphere} \neq 4 : \pi$$

This fundamental overturns the formula in Nine Chapters.

(3) Hypothetical reasoning

The hypothetical reasoning is a commonly used reasoning in mathematical reasoning. First, let us examine how Liu Hui utilized the hypothetical reasoning with a sufficient condition. For example, Liu Hui's commentary to the tomb tunnel [羨除 *yanchu*] procedure in the Work Discussing Chapter [6, p. 184] and [3, pp. 432–437]:

> No [perpendicular] cross-sections of [two solids of the same height] are not equal squares, [yet] a pyramid with a square base and a yang-ma [陽馬 *yangma*] [of the same height and base] are equal in volume.

See Figure 3.

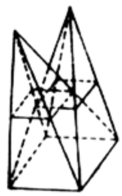


Fig. 3

The statement was written too simplistically to comprehend literally. It can be expanded to be:

If, for each level, the cross-sections of the two 3-dimensional solids of the same height are equal squares (P), then the volumes of the two solids should be equal as well (Q).

For the pyramid with a square base and yang-ma in question, the cross-sections of them for each level are the same squares (P); therefore the volumes of the pyramid with a square base and yang-ma in question should have the same volume (Q).

The reasoning format is, if P then Q; now P, therefore Q. In a true hypothetical reasoning, if the condition P is true, so is the condition Q; if the P is false, then the truth value of Q is uncertain. Liu Hui fully understood this. When we divide a rectangular parallelepiped from one edge towards its opposite, we get two moat-ends [塹堵 *qiandu*] (right triangular prisms).

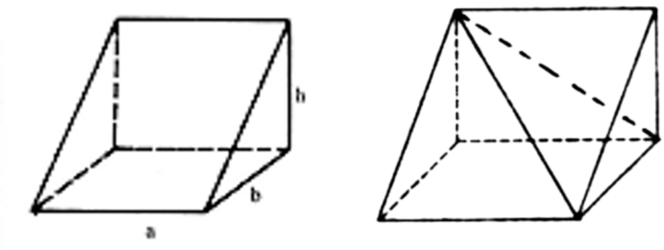


Fig. 4 moat-end [*qiandu*] (left); yang-ma [*yang-ma*] and turtle-forelimb [*bienao*] (right)

When we further divide a moat-end from one vertex towards one of its opposite edges, we get a yang-ma [陽馬] (a pyramid with a square base and four right triangular sides) and a turtle-forelimb [鼈臑 *bienao*] (a tetrahedron with all four sides being right triangles). See the right of Figure 4. Nine Chapters provided the procedure of finding the volume of a moat-end:

> Procedure says: The width and the length [should be] multiplied together; use the height to multiply [the product], then divide [the result] by 2. That is,

$$V_q = \frac{1}{2}abh. \tag{2}$$

And it also provided the volume of a yang-ma:

> Procedure says: The width and the length [should be] multiplied together; use the height to multiply [the product], then divide [the result] by 3. That is,

$$V_y = \frac{1}{3}abh. \tag{3}$$

It provided that of a turtle-forelimb as well:

> Procedure says: The width and the length [should be] multiplied together; use the height to multiply [the product], then divide [the result] by 6. That is,

$$V_b = \frac{1}{6}abh. \tag{4}$$

V_q, V_y, V_b, a, b and h are the respective volume of moat-end, yang-ma, turtle-forelimb, their width, length, and height.[15] Because a cube can be divided into 3 congruent yang-mas or 6 congruent turtle-forelimbs which form three pairs of symmetric ones (See Figure 5), Liu Hui described it as, "Observe how the cube is divided and the shape [of resulting solids] are congruent, then it is easy to verify [these formulas]."

[15] For the procedures of finding the volumes of a moat-end, yang-ma, and turtle-forelimb, see [6, pp. 182–183] and [3, pp. 428–433].

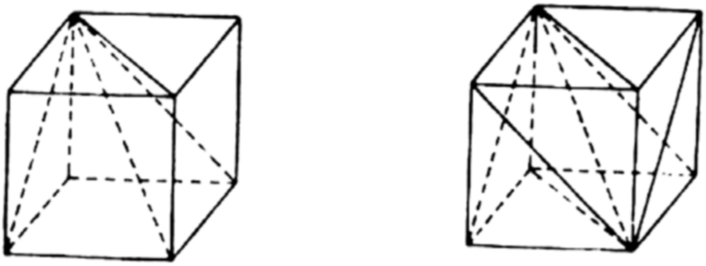


Fig. 5 A cube is divided into three yang-mas, as well as six turtle-forelimbs

That is to say, when the length, width, and height are equal, it is easy to use the geometric models to test and verify that formulas (3) and (4) are correct, as depicted in Figure 5. However, when the length, width, and height are not the same, a rectangular parallelepiped can not be divided into three congruent yang-mas or six turtle-forelimbs. It is impossible to use the testing method to verify formulae (3) and (4), just as Liu Hui commented [6, pp. 182–183] and [3, pp. 430–431]:

> The turtle-forelimbs are of the different shapes, so are the yang-mas. When yang-mas are of the different shapes, then they will not match perfectly. If they do not match perfectly, then it is difficult.

His reasoning style goes as follows:

If the polyhedrons are congruent (P), then their volumes are equal (Q). Now that the polyhedrons are not congruent (P), then it is difficult to tell. (the truth value of Q is uncertain).

That is, to prove formulas (3) and (4) for general cases, one has to find some other ways.

(4) Disjunctive reasoning

Liu Hui used the disjunctive reasoning in many places. In the basic arithmetical calculations, the order of multiplication and division can be switched, "different orders of carrying out multiplications and divisions have their meaning, but give the same result" [6, p. 187] and [3, pp. 442–443]. In the commentary to the 'suppose' procedure in the Millet and Rice Chapter, Liu Hui supported the order of operation to be multiplication first then division because "if [we] divide first and then multiply, there might create fractions [in the process]; that is why the procedure uses the other order." [6, p. 70] and [3, pp. 224–225] This is disjunctive reasoning:

> Either we carry out multiplication first and then division, or division first and then multiplication. Now it is not division first and then multiplication, therefore the order of the operation should be multiplication first and then division.

(5) Dilemma Reasoning

Dilemma Reasoning is a combination of hypothetical and disjunctive reasoning. The major presupposition consists of two hypothetical statements and the minor presupposition is a disjunctive judgment. For example, the style Liu Hui employed in disapproving the incorrect formula for the area of a circle, $S = \frac{1}{12}L^2$ is no other than reasoning by contradiction.

Liu stated [6, p. 23] and [3, pp. 186–187]:

> The ratio of the perimeter of a [regular inscribed] hexagon [inside a circle] to the diameter of the circle is three to one. Therefore, multiplying the perimeter of the hexagon by itself is like nine times the square of the diameter. Nine squares make 12 [inscribed regular] dodecagons. So divide the area of the nine squares to get the area of a dodecagon. Now if [we] make the circumference [of the circle] multiply itself, this is not like nine times the square of the diameter. Then dividing the result by 12 does not yield the area of a dodecagon. If we want to use it as the area of the circle, the discrepancy is too much.

This has two hypothetical presuppositions. One, a one-twelfth of the square of the perimeter of an inscribed regular hexagon (as the circumference of the circle) is the area of an inscribed regular dodecagon inside the circle, which is less than the area of the circle; the other, a one-twelfth of the square of the circumference is greater than the area of the circle. Moreover, there is a disjunctive presupposition: a one-twelfth of the square of the perimeter of an inscribed regular hexagon or a one-twelfth of the square of the circumference. The conclusion is, one quantity is less than and the other is greater than the area of a circle. Both proved that the above formula was incorrect.

Further more, Liu Hui utilized infinite inferences many times, which can be construed as the prototype of a mathematical induction principle.

These analyses present a very small number of many instances of deductive reasoning by Liu Hui; yet it is sufficient to demonstrate that Liu Hui in effect employed several major reasoning formats described in the textbooks of modern formal logic.

3 Liu Hui's Mathematical Proofs

The reasoning statements discussed above, due to the truthfulness of their presuppositions, form de facto proofs or part of their argument. The most beautiful proofs provided by Liu Hui are (1) a proof for the procedure of finding the area of a circle in Nine Chapters and (2) his proof for the Liu Hui Principle proposed by Liu himself.

These two proofs represent two major styles among Liu's proofs: synthesis [綜合 *zonghe*] method and a combination of synthesis and analysis [分析 *fenxi*] methods.[16]

[16] The synthesis [*zonghe*] method refers to the reasoning in which one starts with the given conditions and derive the conclusion while in the analysis [*fenxi*] method, one starts with the conclusion and figure out the condition, say A, needed to obtain the conclusion, and then continues with condi-

1. A proof for procedure of the area of a circle.

Liu believed that the traditional reasoning behind the area of a circle was based on the assumption that the ratio of the circumference of a circle to its diameter is 3 to one, which did not prove the formula properly. Therefore, he proposed a proof based on a limiting process and infinitesimally small sectors produced in the process of circle division. He stated [6, pp. 18–19] and [3, pp. 178–181]:

> Another view, draw the figure: Use one side of an [inscribed regular] hexagon to multiply the radius of the circle. Three times [the product] is the area of an [inscribed regular] dodecagon. If [we] divide the circle further [following the same principle] and use one side of the dodecagon to multiply the radius of the circle. Six times [the product] is the area of an [inscribed regular] 24-gon. The finer we divide the circle to be, the smaller the discrepancy [between the area of the inscribed regular polygon and that of the circle]. Divide the circle until it cannot be divided any more; then [the polygon obtained] will match [perfectly] with the circle and there is no discrepancy. [The line segment from the] outside of [the middle of] one side of the inscribed polygons [to the circumference of the circle] is called excess radius [余径 *yüjing*]. Use perimeter of a regular polygon to multiply its excess radius. Then [the sum of] this area [and that of the regular polygon] exceed [that of] the circle. If the division [of the circle] is so that [the polygon] match perfectly with the circle, then there is no excess radius. If there is no excess radius outside the polygon, then [the sum] of the areas will not surpass that of the circle. Use one side of this [infinite-sided] polygon to multiply the radius, [the area of which is] twice of the sector subtended by one side of the polygon. Hence use one half of the circumference of the circle to multiply the radius to produce the area of the circle.

Liu Hui first took several steps in the limiting process, starting with an inscribed regular hexagon inside a circle. Let us use S_n to denote the area of the inscribed regular 6×2^n-sided polygon obtained after the n-th step of the circle division process and S the area of the circle. Liu Hui demonstrated that

$$S_{n+1} < S < S_n + 2(S_{n+1} - S_n)$$

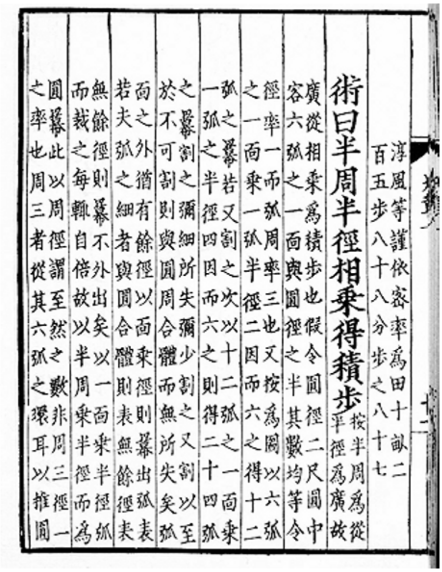
and

$$\lim_{n\to\infty} S_n = S,$$

and

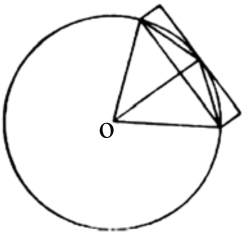
$$\lim_{n\to\infty} [S_n + 2(S_{n+1} - S_n)] = S.$$

tion A and figures out the condition B needed to obtain the condition A, and etc. This reserve-styled argument aims to find the conditions in the reversed order so as to find the condition to match the ones given in the statement of the question.

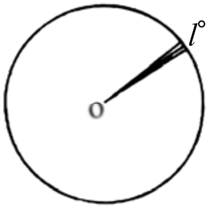

淳風等謹依密率為田十畝二百五步八十八分步之八十七
術曰半周半徑相乘得積步按半周為從半徑為廣故廣從相乘為積步也假令圓徑二尺圓中容六弧之一面與圓徑之半其數均等令徑率一而弧周率三也又按為圖以六弧之一面乘一弧半徑二因而六之得十二弧之冪若又割之次以十二弧之一面乘一弧之半徑四因而六之則得二十四弧之冪割之彌細所失彌少割之又割以至於不可割則與圓周合體而無所失矣弧面之外猶有餘徑以面乘徑則冪出弧表若夫弧之細者與圓合體則表無餘徑表無餘徑則冪不外出矣以一面乘半徑弧而裁之每輒自倍故以半周乘半徑而為圓冪此以周徑謂至然之數非周三徑一之率也周三者從其六弧之數環耳以推圓

(1)

(2)



(3)



(4)

Fig. 6

And Liu Hui considered the inscribed regular infinitely many-sided polygon that matched the circle perfectly and divided it as infinitely many congruent isosceles triangles, each of which has the center of the circle as one vertex, the infinitesimally small "side" as its base, and the radius of the circle as its height. Assume the length of the base is l_i and area of the isosceles is A_i. Then, it is obvious that

$$l_i r = 2A_i.$$

The sum of the length for the base of the infinitely many isosceles triangles is equal to the circumference of the circle; the sum of their areas, the area of the circle:

$\sum_{i=1}^{\infty} A_i = S$. Therefore,

$$\sum_{i=1}^{\infty} l_i r = Lr = 2\sum_{i=1}^{\infty} A_i = 2S.$$

Then we can get S, which is Formula (1) in Section 1 (see Guo [7], [9] and K. Chemla et Guo Shuchun [3]).

This is a complete proof for the procedure of finding the area of a circle in Nine Chapters; moreover, this is a typical synthesis method: From several known conditions, Liu Hui arrived at the conclusion through reasoning.[17] This style was utilized most frequently among the proofs in Liu Hui's commentary.

2. The Proof of the Liu Hui Principle

Even more astonishing than the proof discussed above is how Liu Hui employed the concept of taking the limit and the method of dividing polyhedrons into infinitesimally small pieces to prove the Liu Hui Principle. In order to prove Formulas (3) and (4) of the volumes of a *yangma* [yang-ma] and a *bienao* [turtle-forelimb] rigorously, Liu proposed an important principle, i.e., the Liu Hui Principle:

> Divide a *qiandu* [moat-end] in a slanted manner to obtain a *yangma* and a *bienao*. The *yangma* occupies two [thirds of the volume of the *qiandu*] and the *bienao* one [-third]. This ratio will not change.

That is, in a *qiandu*, the relation

$$V_y : V_b = 2 : 1 \tag{5}$$

remains constant. Obviously, as long as the Liu Hui Principle is proved, formulas (3) and (4) can be obtained by formula (2) for the volume of a *qiandu*. To prove formula (5), Liu Hui stated [6, pp. 182–183] and [3, pp. 430–431]:

> Let *yangma* be the inside of the divided solid and *bienao* the outside. Although a geometric model (rectangular parallelepiped) can have a different length and width, this ratio [of the volumes to be 2:1] remains constant. Even if the body of a parallelepiped is different, the ratio stays the same. That is always the way. Suppose a *bienao* has its width, length, and height all equal to 2 *chi*, divide it into two *qiandu* and *bienao* and use red models [to represent these solids]. Moreover, take a *yangma* so that its width, length, and height also equal to 2 *chi*. Divide it into one cube, two *qiandu*, and two *yangma*. Use black models

[17] Liu Hui's commentary to the circular field method constitutes two distinct discussions on the circle division. One was to prove the formula for the area of a circle in Nine Chapters; the other, to find the circumference-diameter ratio. Before 1970, almost all studies on Liu Hui's commentary to the circular field method never discussed the phrases in Liu's commentary, "Use one side of this [infinite-sided] polygon to multiply the radius, [the area of which is] twice the sector subtended by one side of the polygon. Hence use one-half of the circumference of the circle to multiple the radius to produce the area of the circle. [觚而裁之, 每輒自倍, 故以半周乘半径而為圓冪]" This sentence inadvertently links together the process of taking the limit and that of finding the circumference-diameter ratio. Many scholars described that Liu's process of taking the limit was to find the circumference-diameter ratio. In fact, to find the circumference-diameter ratio did not require the process of taking limits because Liu's circumference-diameter ratio was only an approximate value. That is, his process stops after a finite number of steps.

[to represent these solids]. Combine the red and black models to make a *qiandu*, the width, length, and height of which are all equal to 2 *chi*. Bisect its width and length, and then bisect its height. Combine one red and one black *qiandu* models to make a cube with its height equal to one and the base a square of one. Every two *bienao* models together make one *yangma* model. Take the models on the two sides together to make a cube. The portion that can be made into different kinds of parallelepiped occupied 3 while the portion that can be made into a similar parallelepiped occupied 1. Although the body of a parallelepiped might change according to the models, the ratio remains the same. If the portion that is unknown follows the ratio of 2:1 (for the volumes of divided *yangma* and *bienao*), then the ratio for the whole solid is determined. How can this as a principle be void? Take this argument of numbers to the extreme. Take the remaining portion, the width, length, and the height of which is halved. In this remaining solid, three-fourth of it can be found [by repeating the above argument]. The more halves, the tinier the remaining part becomes. When the process is pushed to the step when the remaining part becomes so fine that is called trifle [微 *wei*], which has no shape, why should we consider the remaining part?

Limited by the geometric models at hand, Liu Hui used a model with $a = b = h = 1$. However, Liu Hui explicitly stated that "Even when the measures of the width, length, and height of the parallelepiped might change, the process of division can be carried out according to the procedure". Therefore, his argument fits the general scenario in which a, b, and h are not equal. We describe the general scenario here. See Figure 4, a *qiandu* can be divided into a *yangma* and a *bienao*. Then the *yangma* can be divided further into one rectangular parallelepiped I, two smaller *qiandu* II, and III, two smaller *yangma* IV and V as in Figure 7 (1); the *bienao* in Figure 4 can be divided further into two smaller *qiandu*, II′ and III′, and two smaller *bienao*, IV′ and V′, as in Figure 7 (2). Obviously the small *qiandu* II and II′, III and III′ can be combined as rectangular parallelepipeds congruent to I, as in Figure 8 (2) and (3). The small *yangma* IV and the small *bienao* IV′ as well as V and V′ can be combined to form two *qiandu* congruent to the small *qiandu* II, III, II′ and III′, which in turn can be combined to form a 4th rectangular parallelepiped congruent to I, as in Figure 8 (4). Obviously, in the first three parallelepipeds, I, II-II′ and III-III′, the ratios of volumes for *yangma* and *bienao* are all equal to 2 : 1.

In fact, as early as 1930's, Japanese scholar Mikami had solved this problem. As for the ways of combining the small *qiendu*'s, he proposed two possibilities, in addition to the aforementioned combining II with II′ and III with III′, it could also be done with II with III and II′ with III′. Mikami was leaning toward the latter [10]. His work did not catch the attentions of Li Yan or Qian Baocong. Therefore, this problem was left by Qian as an open problem to be solved in [11]. In 1980, Hideki Kawahara in his Japanese edition of *The Nine Chapters on the Mathematical Procedures commented by Liu Hui* provided much more detailed discussions on Mikami's conclusions of this problem [12]. Moreover, this question was also discussed independently by Danish scholar D. B. Wagner in 1979 [13] and Guo Shuchun in 1980 [8]. Wagner took the approach of combining II with III and II′ with III′ while Chemla and Guo in [3] II with II′ and III with III′.

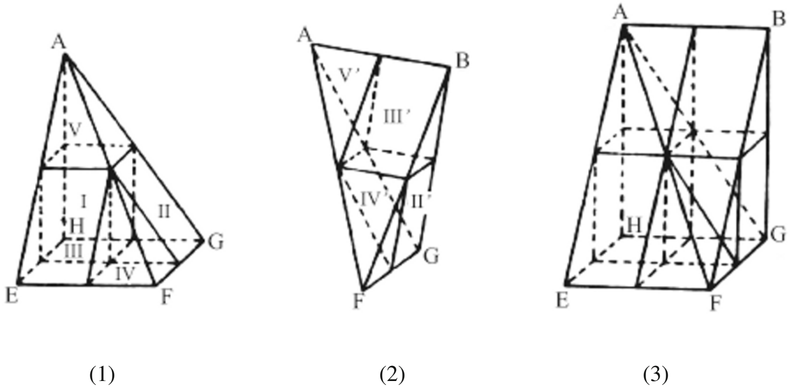


Fig. 7 Decompositions of a *yangma* and a *bienao* which compose a *qiandu*

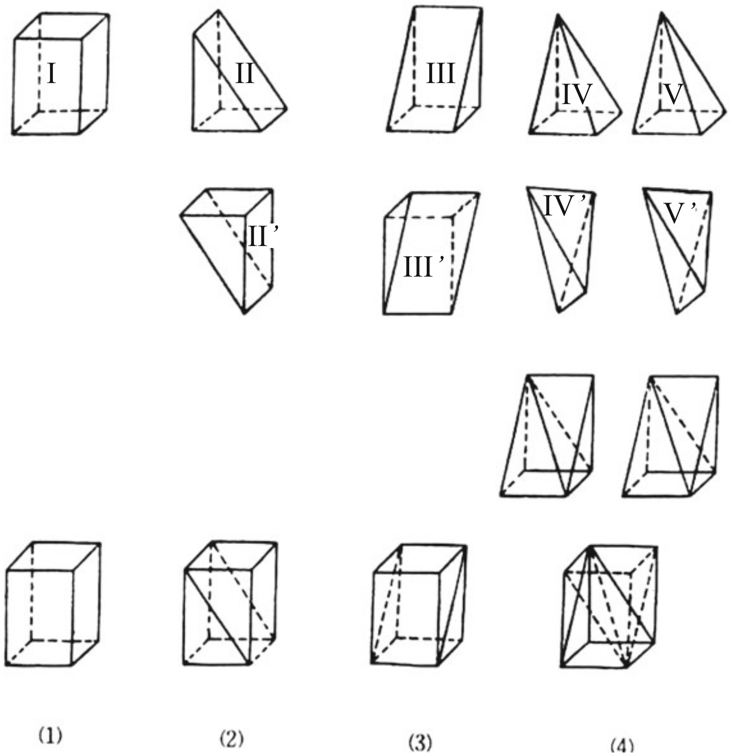
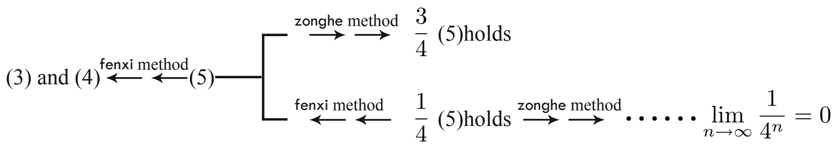


Fig. 8 The proof of the Liu Hui Principle

That is, relation (5) holds in three fourths of the original *qiandu*. Liu Hui contended that if relation (5) can be proved in the 4th rectangular parallelepiped, then (5) is proved for the entire *qiandu*. On the other hand, the two *qiandu* in the 4th rectangular parallelepiped are mathematically similar to the original *qiandu*. Consequently, the process of dividing a *qiandu* discussed above can be applied step-by-step to these two small *qiandu* in the 4th rectangular parallelepiped. Then in $\frac{3}{4}$ of these two *qiandu*, relation (5) holds while it remains to be proved in $\frac{1}{4}$ of them, i.e., $\frac{1}{4} \times \frac{1}{4}$ of the original *qiandu*. This process can be repeated indefinitely. After n divisions, Formula (5) remains to be proved in $\frac{1}{4^n}$ of the original *qiandu*. As well-known, $\lim_{n\to\infty} \frac{1}{4^n} = 0$. As a result, the Liu Hui Principle is proved for the entire *qiandu* [8, pp. 47–62], [9] and [3]. This process can be summarized as



From this diagram, we can see this proof was mainly analysis [*fenxi*], interspersed with synthesis [*zonghe*].

3. Liu Hui's system of mathematical theory

Liu Hui's discussions on fractions, quantities, areas, volumes, and right triangles all constitute a part of his own system of theory. And this system is different from that of Nine Chapters. Take the discussion of the volumes as an example. When discussing his own approach to finding the volume of polyhedrons, Liu Hui stated [6, p. 179] and [3, pp. 422–423]:

> This Chapter deals with *qiandu* and *yangma*, several of which can combine to form a cube. That is why the mathematician sets up three types of geometric models in order to measure the volumes with the height or depth.

At the time of Nine Chapters, the method employed to find the volumes of polyhedrons was the testing method by geometric models. Therefore, the cube, *qiandu* and *yangma*, (see Figure 9) the width, length, and the height of which are all 1 *chi*, held a central position in this approach. The approach to find the volumes of cylinder or circular cone, polyhedrons related to circles, depended on the area of its base. The system of deriving volumes in Nine Chapters can be described in Figure 10.

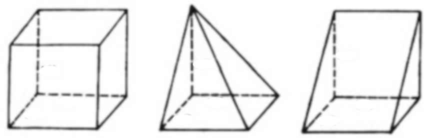


Fig. 9

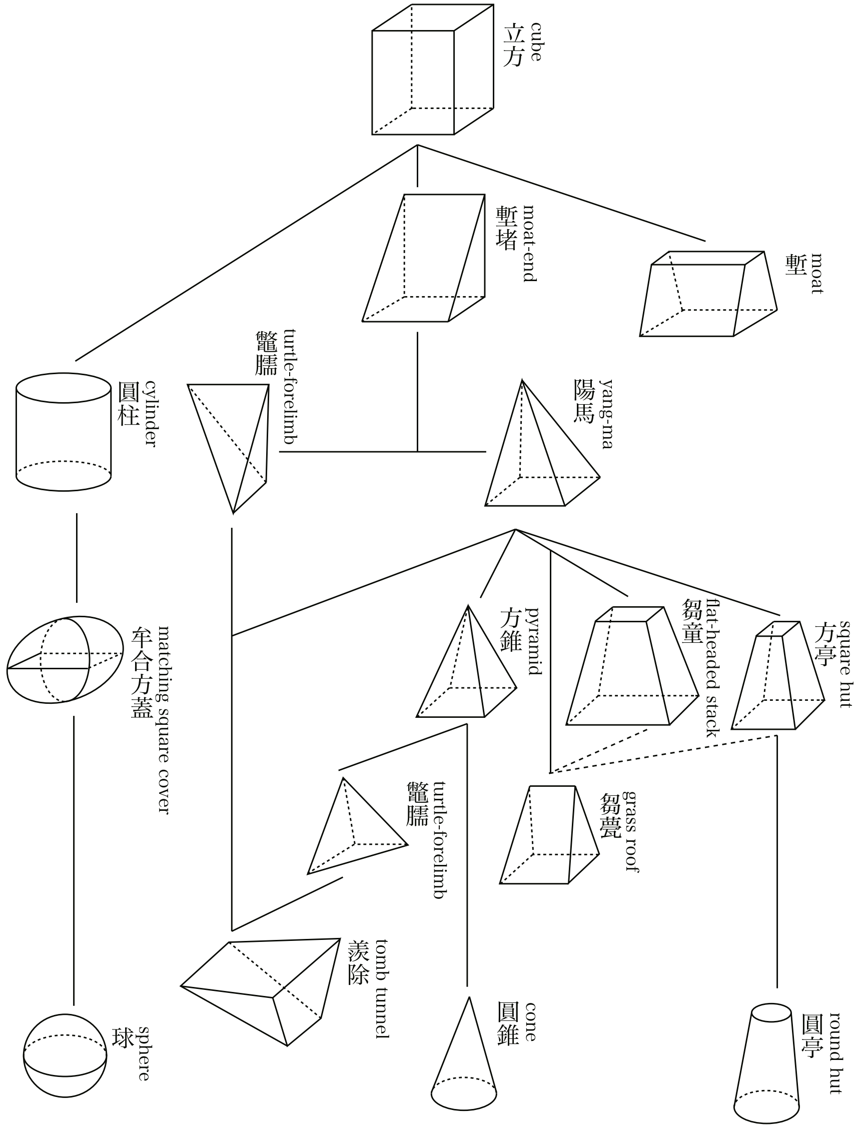

cube
立方
moat-end
塹堵
moat
塹
turtle-forelimb
鼈臑
cylinder
圓柱
yang-ma
陽馬
matching square cover
牟合方蓋
pyramid
方錐
flat-headed stack
芻童
square hut
方亭
turtle-forelimb
鼈臑
grass roof
芻甍
tomb tunnel
羨除
sphere
球
cone
圓錐
round hut
圓亭

Fig. 10

On the other hand, the basis of Liu Hui's theory on the volumes of polyhedrons is the Liu Hui Principle. After completing the proof of this principle, Liu Hui stated [6, p. 183] and [3, pp. 432–433]:

> Without *bienao*, [we] cannot verify the volume of *yangma*; without *yangma*, [we] cannot verify the solids similar to cones and frustums. Therefore, *bienao* is the main reason for finding the volumes of many polyhedrons.

Liu Hui contended that *bienao* played a key role in solving problems of volumes of polyhedrons. For many polyhedrons, their volumes can be obtained, through finitely many steps, by dividing them into a collection of rectangular parallelepipeds, *qiandu*, *yangma*, and *bienao*; since the formulas for whose volumes were established and proved, by adding these volumes together, one can find the volume of the solid in question.

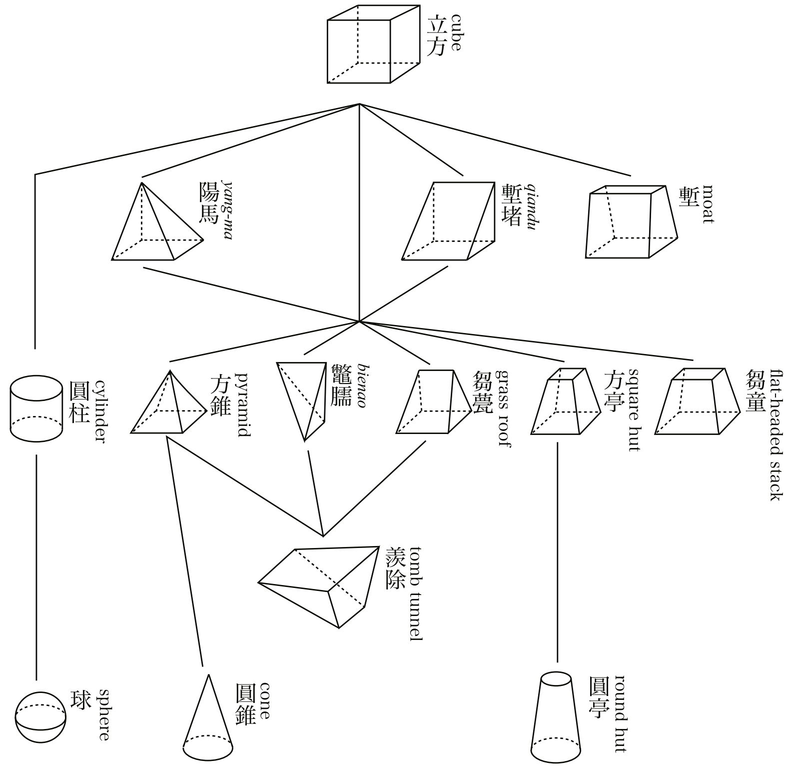


Fig. 11 Liu Hui's relation among geometric solids[18]

As for the volumes of solids related to circles (i.e. circular cylinder or sphere), their volumes can be obtained by comparing the area of cross-sections. The system of Liu Hui's theory for volumes can be described in Figure 11. Liu Hui viewed

[18] The downward lines in Fig.11 reflect the fact that the solids below were derived from the ones above.

bienao as the smallest unit in dividing polyhedrons. Both this concept of dividing volumes into smaller pieces and the procedure of finding the volume of a *bienao* have to depend on the realization of the infinitesimal division. That is, Liu's theory of the volumes for the polyhedrons in effect established on the concept of infinitesimal division, which is surprisingly in line with the modern theory of volumes. The great mathematician Gauss (1777–1855) conjectured that the solutions to finding the volumes of polyhedrons cannot be achieved without carrying out divisions of the solids into infinitesimal small pieces. Based on this conjecture, Hilbert (1862–1943) proposed the third problem in his address of 1900 to the international congress of mathematicians [2, pp. 60–84]. His follower Dehn (1878–1952) soon afterwards provided a positive answer. Liu Hui in the third century started to consider problems considered by mathematicians in the 19th and 20th centuries.

Since some years ago, the different branches in mathematics have been depicted as a tree. Located at the roots are algebra, plane geometry, trigonometry, analytic geometry, and irrational numbers. From these roots grow the strong trunk, differential and integral calculus. On the top of the trunk sprout many branches including the branches of higher mathematics [1, p. 491]. In fact, 1700 years ago, Liu Hui had the concept of the mathematical tree. He stated:

> Things of various kinds can be used to find each other, each of which can be found its location in the complex relations. The reason that the branches are apart but share the same trunk is that they came from the same root. Analyze the principles by virtue of verbal formulation; explain the substance of things using figures in the hope of achieving simplicity while remaining complete and general but not obscure, so that the reader [of the commentary] will be able to grasp more than half [6, p. 1] and [3, pp. 126–127].[19]

Liu Hui's mathematical tree starts from a point. What is this starting point? Liu Hui stated [6, pp. 1–2] and [3, pp. 126–129]:

> Although they were described as Nine Branches in Mathematics [九数 *Jiushu*], they can exhaust the fine details, get into extremely small matters, and explore and measure without any bound. As for disseminating the methods, it is just like try square and compass [規矩 *guiju*] and measurements [度量 *duliang*] can be used to find the commonality; therefore, they are not extremely difficult.

The term try square and compass represents the figures and diagrams in the space while the term measurements the relations among measured quantities. That is to say, mathematical methodology from generation to generation is the unification of geometric problems and relations among the measured quantities in the objective world. The *guiju* and *duliang* can be seen as the root of Liu Hui's tree of mathematics. Methods in mathematics are born out of the *guiju* and *duliang*. This also reflects the characteristic of ancient Chinese mathematics—the union of shapes and numbers as well as that of geometric problems and arithmetical algebraic methods.

[19] The translation of last sentence was taken from Martzloff [4, pp. 69–70].

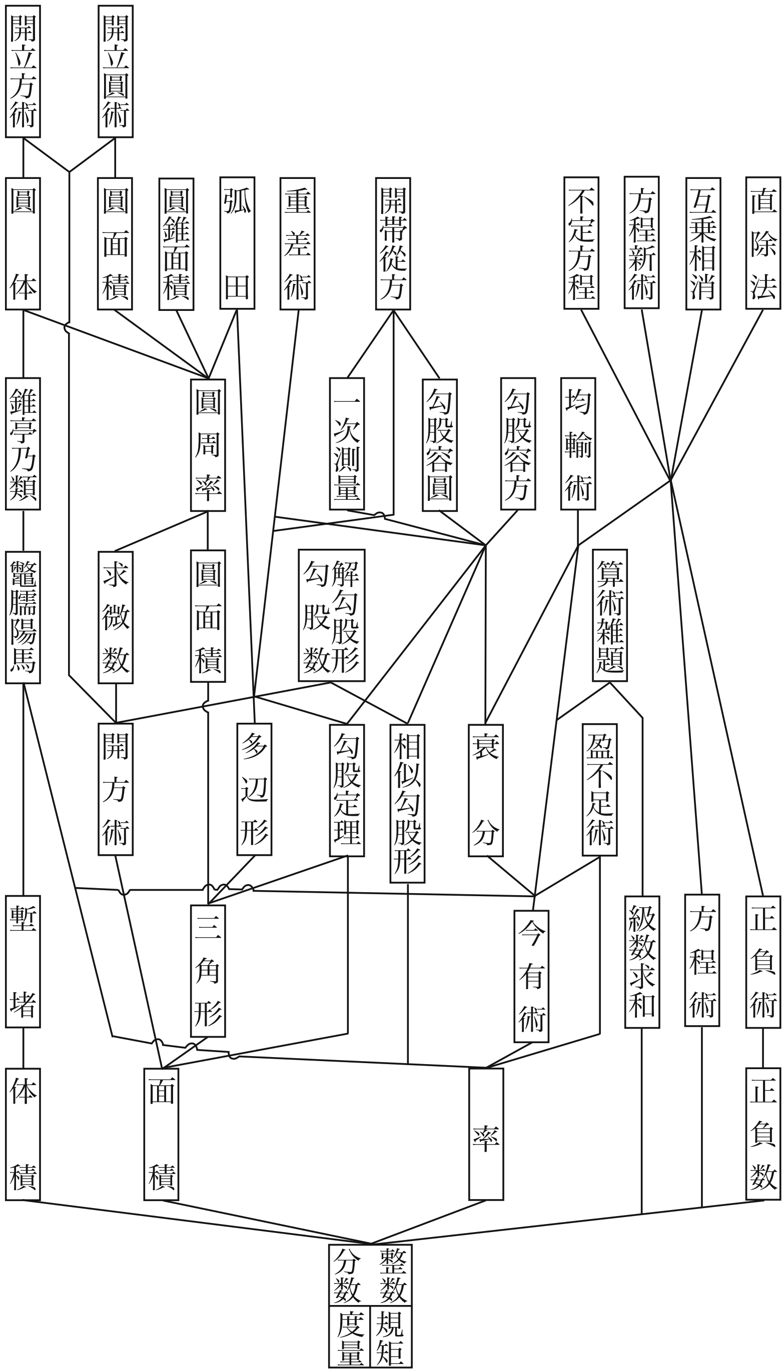
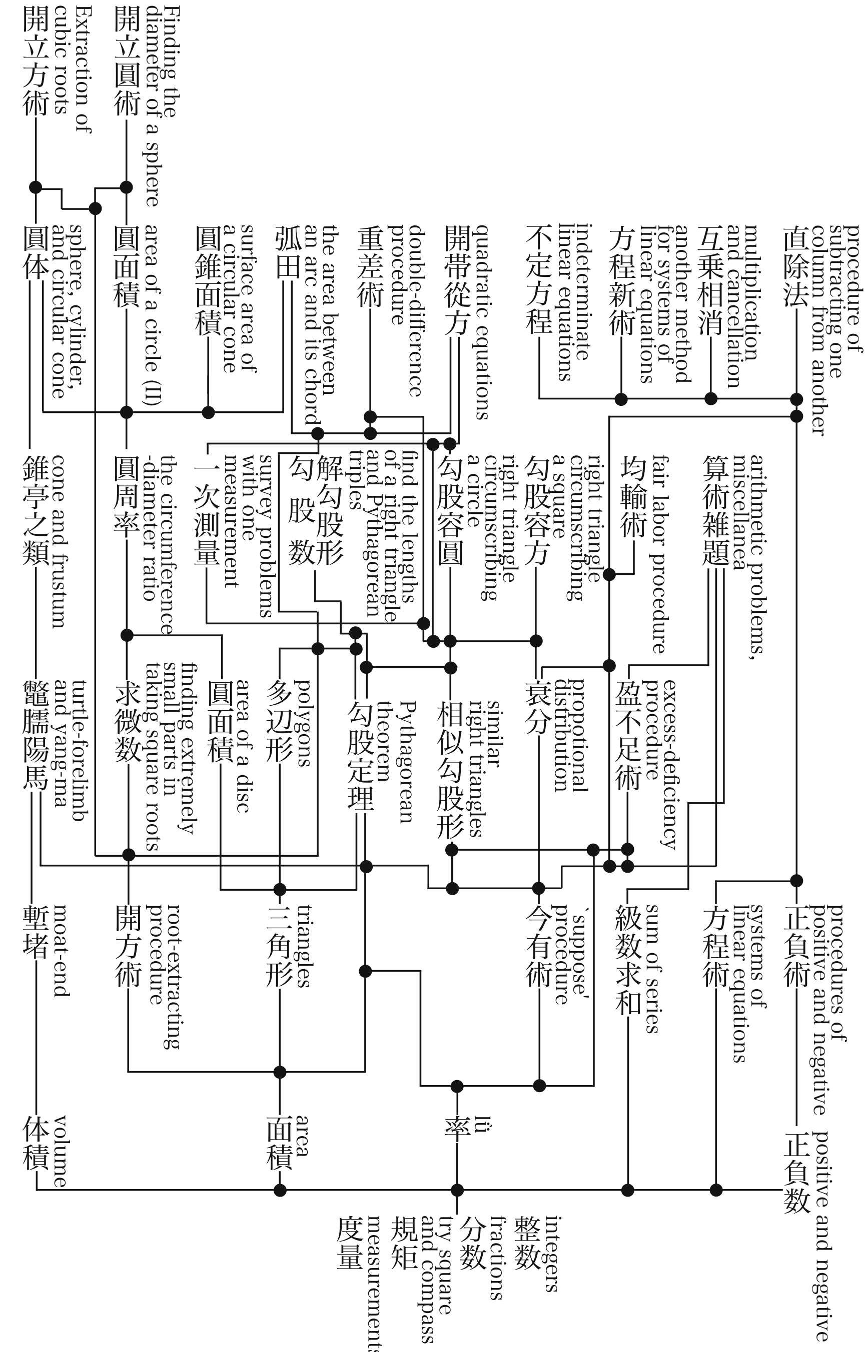


Fig. 12 Liu Hui's mathematical tree


直除法
procedure of subtracting one column from another
互乘相消
multiplication and cancellation
方程新術
another method for systems of linear equations
不定方程
indeterminate linear equations
開帯從方
quadratic equations
重差術
double-difference procedure
弧田
the area between an arc and its chord
圓錐面積
surface area of a circular cone
圓面積
area of a circle (II)
圓体
sphere, cylinder, and circular cone
開立圓術
Finding the diameter of a sphere
開立方術
Extraction of cubic roots
算術雜題
arithmetic problems, miscellanea
均輸術
fair labor procedure
勾股容方
right triangle circumscribing a square
勾股容圓
right triangle circumscribing a circle
解勾股形
find the lengths of a right triangle and Pythagorean triples
勾股数
一次測量
survey problems with one measurement
圓周率
the circumference-diameter ratio
錐亭之類
cone and frustum
盈不足術
excess-deficiency procedure
衰分
propotional distribution
相似勾股形
similar right triangles
勾股定理
Pythagorean theorem
圓面積
area of a disc
多辺形
polygons
求微数
finding extremely small parts in taking square roots
鼈臑陽馬
turtle-forelimb and yang-ma
正負術
procedures of positive and negative
方程術
systems of linear equations
級数求和
sum of series
今有術
'suppose' procedure
三角形
triangles
開方術
root-extracting procedure
塹堵
moat-end
正負数
positive and negative
率
lü
面積
area
体積
volume
整数
integers
分数
fractions
規矩
try square and compass
度量
measurements

Liu Hui's tree came out of the two roots, measuring tools and measurements. They are unified under numbers, upon which grows the trunk of computations of quantities. Based on the unproved yet agreed-upon formulas of the rectangular area and volume of a rectangular parallelepiped and the definition of ratio [率 *lü*], there grows the branches of the four arithmetic operations for integers and fractions, the 'suppose' procedure, proportional distribution procedure, fair labor procedure, excess-deficiency procedure, root-extracting procedure, rectangular array procedure, solutions to area and volumes, and at least the solution to right triangles and survey, from which grows more refined branches of all kinds of mathematical methods. Eventually, all these contribute to form a leafy tree full of fruits, as depicted in Figure 12.

Liu Hui's system of mathematics is "achieving simplicity while remaining complete and general but not obscure [約而能周, 通而不黷]." That is, it is simple but complete, far-reaching without obstacles. Even though in the form of a commentary, Liu Hui cannot but separate his mathematical knowledge into the algorithms and questions in Nine Chapters. It is worth mentioning that his commentary did not contain any self-conflict paradox logically. This shows the level of his logical reasoning. Liu Hui's mathematical system was developed upon the frame of the mathematics in Nine Chapters. It inherited the correct content in Nine Chapters; moreover, it molded and complemented it. In short, Liu Hui's commentary, compared with the texts in Nine Chapters, transformed the quality of the mathematical content in it.

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