Math 22a: Vector Calculus and Linear Algebra

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Abstract

This course covers multivariable calculus and linear algebra for students interested in theoretical sciences. It covers the same topics as Math 21 but with more rigor. Students are taught techniques of proof and mathematical reasoning. The workload and content is comparable with the 21 sequence. But unlike in the latter, the linear algebra and calculus is more interlinked. Students in 22 will be taught some proof aspects.

1. Week: September 4 to 6, 2018

1. Unit: Pythagoras Theorem, Sep 4, 2018

We introduce **matrices** and **vectors**, the **matrix multiplication**, the **trace** and the **dot product**. After proving the **Cauchy-Schwartz inequality**, we can define **lengths** and **angles**, get the cos-**formula** and prove the **Pythagoras theorem.** The addition and scalar multiplication leads to the concept of linear space and coordinates. In a homework, we see that the dot product $\mathrm{tr}(A^T B)$ between $n \times m$ matrices generalizes the dot product of vectors. It is the power of mathematics which allows to prove something like Pythagoras and apply it in situations which go beyond the original scope because the mathematical structures are the same.

2. Unit: Row reduction, Sep 6, 2018

Matrices naturally appear when solving **systems of linear equations** $Ax = b$. Such systems can have either zero, one or infinitely many solutions. We introduce this here already as in calculus, such equations will appear when finding maxima or minima using the Lagrange method. We prove that the row reduced echelon form is unique. An effective way to solve equations is to row reduce the augmented coefficient matrix and use the kernel of A to parametrize the homogeneous solutions. In the homework we explore the process of row reduction both theoretically as well as practically.

3. Unit: Proof seminar: Induction

2. Week: September 11 to 13, 2018

4. Unit: Area and Volume Sep 11,2018

By computing kernels $Ax = 0$, we can construct the **orthogonal complement** of a subspace defined by the rows of some matrix. A special case is the **cross product** $v \times w$ between two vectors in $\mathbb{R}^3$. It leads to the **triple scalar product** $u \cdot (v \times w)$. We look at constructions in geometry and finding distance formulas like the distance between two lines. We also learn how to **measure area** and **volume**. In a homework, we see that the cross product can be written as the **anti-commutator** of matrices which so leads to a way to generalize the cross product also to higher dimensions.

5. Unit: Surfaces, Sep 13, 2018

We first look at **affine planes** $a \cdot x = c$, **quadrics** $(x, Ax) = c$ and then at more general surfaces defined implicitly by $g(x, y, z) = 0$. We mostly look at three dimensional cases, cover level surfaces and especially quadrics $\vec{x}A\vec{x} = c$ defined by a symmetric matrix A. We explore the classification of quadratic surfaces in small dimensions. We will later produce a classification using the spectral theorem in linear algebra. The four dimensional case is important for three reasons: important **computer vision algorithms** use four dimensional space to deal with projective transformations. **Space-time** is four dimensional and the light cone is a cone in four dimensional space. Third, the quaternion space is four dimensional. The unit sphere in the quaternions is important as it is the only sphere besides the circle, on which one can multiply.

6. Unit: Proof seminar: Visual proofs

3. Week: September 18 to 20, 2018

7. Unit: Curves, Sep 18, 2018

We define curves by **parameterization** $\vec{r}(t)$, where the **parameter** t is in some **parameter interval**. The parametrization gives a dynamical and constructive description which describes how the curve is traced if t is interpreted a **time variable**. Differentiation of a parametrization $\vec{r}(t)$ leads to the **velocity** $\vec{r}\,'(t)$, a vector which is tangent to the curve at $\vec{r}(t)$. Acceleration etc is defined the same. We also learn how to get from $\vec{r}''(t)$ and $\vec{r}\,'(0)$ and $\vec{r}(0)$ the position $\vec{r}(t)$ by integration. A special case is the **free fall**, where the acceleration vector is constant. We also introduce the **normal vector** $\vec{N}$ and **bi-normal vector** $\vec{B}$ to get the **Frénet frame**. Finally, we look a curvature. The **curvature** $\kappa(t)$ of a curve measures how much a curve is bent. Both acceleration and curvature involve second derivatives, but curvature is an intrinsic quantity which does not depend on parameterizations. We verify the formula $\kappa(t) = |T'(t)|/|r'(t)| = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|r\,'(t)|$, where $T(t) = r'(t)$ is the **unit normal vector**.

8. Unit: Riemann integration and arc length Sep 20, 2018

In this lecture we take the opportunity to properly define the Riemann integral and prove that a continuous function is Riemann integrable. The **arc length** of a rectifiable curve is defined as a limiting length of polygons. It leads to the **arc length** integral $\int_a^b |r'(t)| \, dt$. We verify that a re-parametrization of a curve does not change the arc length.

9. Unit: Proof seminar: intuition in mathematics Sep 22, 2018

4. Week: September 25-27, 2018

10. Unit: Other coordinates, Sep 25, 2018

In this lecture we introduce various coordinate systems. This includes **polar coordinates** (r, θ) and **spherical coordinates** $x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta), y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi)$. We also take the opportunity to define complex numbers. In order to understand coordinate changes, we need the notion of the **Jacobian matrix**, which serves as the derivative. The computation of length, area or volume in other coordinate systems can be understood in terms of the Jacobian matrix. The volume distortion factor is the determinant. In a homework, we explore other coordinate systems like **toral coordinates** or spherical coordinates in four dimensions which are related to **quaternions**.

11. Unit: Parametric surfaces, Sep 27, 2018

We learn how to get from implicit descriptions like $x^2 + y^2 + z^2 - 1 = 0$ to parametrizations like $\vec{r}(\theta, \phi) = [\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)]$. In many cases, it is possible to switch from a parametric description to an implicit and back. Examples are the plane, spheres, graphs of functions of two variables or **surfaces of revolution**. It is important to have an example for each of these 4 basic classes of surfaces. Using a computer, one can **visualize** complicated surfaces. In computer applications, the parameterization $\vec{r}(u, v)$ is called the **"uv-map"**. We will use eigenvectors to parametrize a general quadric $xAx = 1$ with positive definite A.

12. Unit: Proof seminar: Creativity Sep 29, 2018

5. Week: October 2-4, 2018, Oct 2, 2018

13. Unit: First exam

We have our first exam in class. Actually in Hall E. There will be a review on Sunday, September 30 at 7 PM in Hall E.

14. Unit: Partial derivatives, October 4, 2018

We look at notions of continuity and notions of differentiability. As this is historically one of the major difficulties when developing calculus, it is important to see the difficulty in examples, both in one and in higher dimensions. While for functions of one variable, continuity can fail with jump discontinuities, infinities or singular oscillations, in higher dimensions, this is more interesting. Examples like $f(x, y) = (x^3 - y)/(x^2 + y^2)$ show this. Discontinuities appear naturally, when looking at the position of minima of a function of several variables. The disappearance can lead to **catastrophes**, discontinuous changes of the minimum.

15. Unit: Proof seminar: Techniques Oct 6, 2018

6. Week: October 9-11, 2018

16. Unit: Partial differential equations, Oct 9, 2018

We illustrate these derivatives with various **partial differential equations** like heat, wave, transport or Laplace equations. These systems are not only at the heart of other sciences, they allow to explain what the partial derivatives mean and how they are important in real life situations. A more challenging homework problem is to describe some of these differential equations in other coordinates. We might explore this in a homework.

17. Unit: Taylor approximation, Oct 11, 2018

We first look at the **linearization** L of a function f, then the **quadratic approximation** Q and finally at the **Taylor theorem** in multiple dimensions. Having seen some partial differential equations already we can use the **transport equation** $u_t = u_x = Du$ which has the solution $u(x + t)$ agreeing with $e^{Dt}u$ which is also a solution. When expanding the exponential, we get the Taylor series. In higher dimensions, the formula $u(x + tv) = u(x) + tD_v(u) \cdot v + D_v^2(u)/2$ is proven in the same way using the **directional derivative** $D_v f = Df \cdot v$. Since $D_v^2(u) = \nabla f v \cdot v$ etc, this gives the Taylor formula.

18. Unit: The exponential, Oct 11, 2018

7. Week: October 16-18, 2018

19. Unit: Chain rule, Oct 16, 2018

The multivariable chain rule is related to directional derivatives and tangent spaces as well as to the implicit function theorem. The **chain rule** $d/dt f(g(t)) = f'(g(t))g'(t)$ from single variable calculus has the same form in higher dimensions. We just have to use the notion of **Jacobians** introduced earlier. As an illustration, we can look at the iteration of maps $x_0 = x, x_1 = f(x), x_2 = f(f(x)), x_3 = f(f(f(x)))), \dots x_n = f^{(n)}(x)$ and have a look the derivative of the n'th iterate $f^{(n)}$. We get so a product of matrices $A_n \cdots, A_1 A_x$ where $A_k(x) = Df(x_k)$. At fixed points, we are led to the concept of eigenvalues seen in the next lecture. We also look at implicit differentiation and the implicit function theorem.

20. Unit: Eigenvalues, Oct 18, 2018

We introduce **eigenvalues** and **eigenvectors** for small matrices. We illustrate it with concrete examples like like reflections, projections, shears or rotations. We motive the concept using **discrete dynamical systems** introduced in the last lecture. An important example is given by **Markov chains**. The main reason to introduce eigenvalues early here is to be able to understand better extrema of functions of two and more dimensions. In a homework, we will look at the concept of page rank as well as the relevance of the maximal eigenvector of a stochastic matrix.

21. Unit: Properties of eigenvalues, Oct 11, 2018

8. Week: October 23-25, 2018

22. Unit: Extrema, Oct 23, 2018

In this lecture, we **maximize and minimize** functions of several variables. First we look at the problem in two dimensions and see that the structure of the eigenvalues is the key for the classification. To find **extrema** of functions $f(x, y)$ of two variables, we look at **critical points**, where the gradient $\nabla f(x, y)$ vanishes. The **discriminant** $D = f_{xx} f_{yy} - f_{xy}^2$ is the determinant of Hessian matrix and so the product of the two eigenvalues. The sign of these eigenvalues determines now whether we have **local maxima**, **local minima** or a **saddle points**. Also for functions $f(x, y, z)$ of several variables, the eigenvalues of the Hessian determine what type of critical point we have. If the Hessian matrix is positive definite meaning that all eigenvalues are positive, then we have a minimum. If all eigenvalues are negative, we have a local maximum.

23. Unit: Lagrange, Oct 25, 2018

In this lecture, we maximize and minimize functions under constraints, also in higher dimensions. This is the **Lagrange frame work** which is important in economics or physics. In order to maximize or minimize a function $f(\vec{x})$ in the presence of a **constraint** $g(\vec{x}) = c$ we need the deriatives of f and g to be parallel. This leads to a system of equations called the **Lagrange equations** $\nabla f = \lambda \nabla g, g = 0$. We can also extremize functions $f(\vec{x})$ having more constraints in which case we have more Lagrange multipliers. We see applications in economics and physics like deriving the Gibbs equations. In a homework, we also see that the eigenvalue equation $Av = \lambda v$ as a Lagrange problem.

24. Unit: Proofs about Extrema, Oct 25, 2018

9. Week: Oct 30 - Nov 1, 2018

25. Unit: Integration, Oct 30, 2018

We learn how to integrate in several variables, also using change of coordinates and by changing the order of integration. We introduce the **Riemann integral**, cover **Fubini's theorem** on a rectangular region and then for more general regions. Double integrals define **area** if $f(x, y) = 1$, triple integrals define **volume** if $f(x, y, z) = 1$. We also look at integrals in other coordinate systems.

26. Unit: Vector fields, Oct 30, 2018

We introduce vector fields and their use in biology (like population model systems), in physics (the analysis of mechanical systems, or velocity fields in weather maps) or in geometry like gradient fields. We look in particular at examples in two or three dimensions. We learn how to **match vector fields** with formulas and introduce **flow lines**, parametrized curves $\vec{r}(t)$ for which the vector $\vec{F}(\vec{r}(t))$ is parallel to $\vec{r}\,'(t)$ at all times.

27. Unit: TBD Oct 25, 2018

10. Week: November 6-8, 2018

28. Unit: Second midterm, Nov 6, 2018

We have the second midterm in class (Hall E)

29. Unit: Line integrals, Nov 8, 2018

For a parametrized curve $\vec{r}(t)$ and a vector field $\vec{F}$, define the **line integral** $\int_C F(\vec{r}(t))\ \vec{r}\,'(t)\ dt$ along a curve in the presence of a vector field. An important example is the case if $\vec{F}$ is a **force field** and where line integral is **work**. In two dimensions, the curl of a field $\text{curl}([P, Q]) = Q_x - P_y$ measures the **vorticity** of the field and if this is zero, the line integral along a simply connected region is zero. For conservative fields $F = \nabla f$, we have the formula $\int_C \nabla f\ ds = f(b) - f(a)$ called **fundamental theorem of line integrals**. In a homework we explore what the relation between the conservative property, zero curl $\text{curl}(\vec{F}) = Q_x - P_y = 0$ and **simply connectedness** is.

30. Unit: TBD Nov 10, 2018

11. Week: November 13-15, 2018

31. Unit: Green theorem, Nov 13, 2018

In this lecture, we cover **Greens theorem** and prove it. Green's theorem as a prototype for Stokes theorem and has many applications. The theorem equates the line integral along a boundary curve C with a double integral of the curl inside the region G: $\int\int_G \text{curl}(\vec{F})(x, y)\ dxdy = \int_C \vec{F}(\vec{r}(t))\ \vec{r}\,'(t)\ dt$. The theorem is useful to **compute areas**: take a field $\vec{F} = [0, x]$ which has constant curl 1. It also allows to compute complicated line integrals. Greens theorem implies that if $\text{curl}(F) = 0$ everywhere in the plane, then the field has the **closed loop property** and is therefore conservative.

32. Unit: exterior derivatives, Nov 15, 2018

We look at three derivatives gradient, curl and divergence and see how this can be unified under the notion of **exterior derivative**. The **curl** of a vector field $\vec{F} = [P, Q, R]$ in three dimensions is a new vector field which can be computed as $\nabla \times \vec{F}$. The three components of $\text{curl}(F)$ are the vorticity of the vector field in the x,y and z direction.

33. Unit: TBD Nov 17, 2018

Thanksgiving

12. Week: November 27-29, 2018

34. Unit: Stokes theorem, Nov 27, 2018

Given a surface S and a fluid with velocity $\vec{F}(x,y,z)$ at the point (x,y,z). The amount of fluid which passes through the membrane S in unit time is called the **flux**. This integral is $\int\int_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \; dudv$. The angle between $\vec{F}$ and the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$ determines the sign of $\vec{dS} = \vec{F} \cdot \vec{n} \; dudv$. Stokes theorem equates the line integral along the boundary C of the surface S with the flux of the "curl" of the field through the surface: $\int_C \vec{F} \; dr = \int\int_S \text{curl}(F) \dot{d}S$. The theorem has application in electromagnetism and explains why the line integral of an curl-free vector field field along a closed curve in space is zero if the field is defined in a simply connected region.

35 Unit: Divergence theorem, Nov 29, 2018

The **divergence** of a vector field in the interior of a solid E is is equal to the flux of the vector field $\vec{F}$ through the boundary. This is true in arbitrary dimensions. The theorem equates the integral $\int\int\int_E \text{div}(\vec{F}) \; dV$ of a vector field $\vec{F}$ with the flux $\int\int_S \vec{F} \cdot \vec{dS}$ through the boundary surface S of E. It is useful for example to compute the gravitational field inside a solid like the sphere or a hollow sphere. The theorem allows also to express the volume of a solid in terms of the boundary. The result holds in any dimension.

13. Week: December 4, 2018

36 Unit: Integral theorems, Dec 4, 2018

In this last lecture of the fall semester, we look at the general structure of integral theorems and indicate what differential forms are. We give an overview over all the integral theorems. In d-dimensions, there are d theorems. We wrap up the entire course.

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This is a preparation draft. The final version with proof seminars is online