LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 2: Gauss-Jordan elimination

Lecture

2.1. If a $n \times m$ matrix A is multiplied with a vector $x \in \mathbb{R}^m$, we get a new vector Ax in $\mathbb{R}^n$. The process $x \to Ax$ defines a **linear map** from $\mathbb{R}^m$ to $\mathbb{R}^n$. Given $b \in \mathbb{R}^n$, one can ask to find x satisfying the **system of linear equations** $Ax = b$. Historically, this gateway to **linear algebra** was walked through much before matrices were even known: there are Babylonian and Chinese roots reaching back thousands of years. [1]

2.2. The best way to solve the system is to **row reduce** the **augmented matrix** $B = [A|b]$. This is a $n \times (m+1)$ matrix as there are $m+1$ columns now. The **Gauss-Jordan elimination** algorithm produces from a matrix B a **row reduced matrix** $\text{rref}(B)$. The algorithm allows to do three things: **subtract a row from another row**, **scale a row** and **swap two rows**. If we look at the system of equations, **all these operations preserve the solution space**. We aim to produce **leading ones** ①, which are matrix entries 1 which are the first non-zero entry in a row. The goal is to get to a matrix which is in **row reduced echelon form**. This means: A) every row which is not zero has a leading one, B) every column with a leading 1 has no other non-zero entries besides the leading one. The third condition is C) every row above a row with a leading one has a leading one to the left.

2.3. We will practice the process in class and homework. Here is a theorem

> **Theorem:** Every matrix A has a unique row reduced echelon form.

Proof. [2] We use the method of **induction** with respect to the number m of columns in the matrix. The **induction assumption** is the case $m = 1$ where only one column exists. By condition B) there can either be zero or 1 entry different from zero. If there is none, we have the zero column. If it is non-zero, it has to be at the top by condition C). We are in row reduced echelon form. Now, let us assume that all $n \times m$ matrices have a unique row reduced echelon form. Take a $n \times (m+1)$ matrix $[A|b]$. It remains in row reduced echelon form, if the last column b is deleted (see lemma). Remove the last column and row reduce is the same as row reducing and then delete the last column. So, the columns of A are uniquely determined after row reduction. Now note that for a row of $[A|b]$ without leading one at the end, all entries are zero so that also

[1] For more, look at the exhibit on the website: google "catch 22 Harvard" to get there

[2] The proof is well known: i.e. Thomas Yuster, Mathematics Magazine, 1984

the last entries agree. Assume we have two row reductions $[A'|b']$ and $[A'|c']$ where A' is the row reduction of A. A leading $\textcircled{1}$ in the last column of $[A'|b']$) happens if and only if the corresponding row in A was zero. So, also $[A', c']$ has that leading $\textcircled{1}$ at the end. Assume now there is no leading one in the last column and $b'_k \neq c'_k$. We have so x, a solution to the equation $A'_{kq}x_q + A'_{k,q+1}x_{q+1} + ...A'_{k,m}x_m = b'_k$. Since solutions to equations stay solutions when row reducing, also $A'_{kq}x_q + A'_{k,q+1}x_{q+1} + \cdots + A'_{k,m}x_m = c'_k$. Therefore $b'_k = c'_k$. □

2.4. A separate lemma allows to break up a proof:

Lemma: If $[A|b]$ is row reduced, then A is row reduced.

Proof. We have to check the three conditions which define row reduced echelon form. □

2.5. It is not true that if A is in row reduced echelon form, then any sub-matrix is in row reduced echelon form. Can you find an example?

Examples

2.6. To row reduce, we use the three steps and document on the right. To save space, we sometimes report only after having done two steps. We circle the **leading $\textcircled{1}$**. Note that we did not immediately go to the leading $\textcircled{1}$ by scaling the first. It is a good idea to **avoid fractions** as much as possible.

$\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 20 & 30 & 40 & 50 & 60 \\ 1 & 2 & 3 & 4 & 4 \end{bmatrix}$	$\to$ row 3 $\to$ row 1	$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 20 & 30 & 40 & 50 & 60 \\ 3 & 4 & 5 & 6 & 7 \end{bmatrix}$	$*1/10$
$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{bmatrix}$	 $-R_1$ $-R_2$	$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	 $-R_1$ $-R_2$
$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$+2R_2$ $*(-1)$	$\begin{bmatrix} \textcircled{1} & 0 & -1 & -2 & -3 \\ 0 & \textcircled{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	

2.7. Finish the following **Suduku problem** which is a game where one has to fix matrices. The rules are that in each of the four 2×2 sub-squares, in each of the four rows and each of the four columns, the entries 1 to 4 have to appear and so add up to 10 $\begin{bmatrix} 2 & 1 & x & 3 \\ 3 & y & z & 1 \\ 4 & 3 & a & 2 \\ b & c & d & e \end{bmatrix}$. We have the equations $2 + 1 + x + 3 = 10, 3 + y + z + 1 = 10, 4 + 3 + a + 2 = 10, b + c + d + e = 10$ for the rows, $2 + 3 + 4 + b = 10, 1 + y + 3 + c = 10, x + z + a + d = 10, 3 + 1 + 2 + e + 10$ for the columns and $2 + 1 + 3 + y = 10, x + 3 + z + 1 = 10, 4 + 3 + b + c = 10, a + 2 + d + e = 10$ for the four squares. We could solve the system by writing down the corresponding augmented matrix and then do row reduction. The solution is $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

Illustrations

The system of equations

$$\left| \begin{array}{ccccccccccccc} x & & & & & + & u & & & & & = & 3 \\ & & y & & & & & + & v & & & = & 5 \\ & & & & z & & & & & + & w & = & 9 \\ x & + & y & + & z & & & & & & & = & 8 \\ & & & & & & u & + & v & + & w & = & 9 \end{array} \right|$$

is a **tomography** problem. These problems appear in **magnetic resonance imaging**. A precursor was was X-ray Computed Tomography (CT) for which Allen MacLeod Cormack got the Nobel in 1979 (Cormack had a sabbatical at Harvard in 1956-1957, where the idea hatched). Cormack lived until 1998 in Winchester MA. He originally had been a physicist. His work had tremendous impact on medicine.

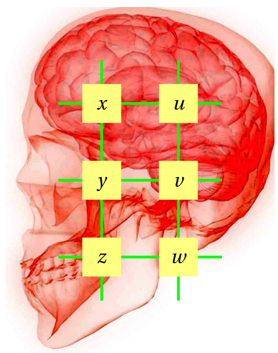

Figure 1. A MRI scanner can measure averages of tissue densities along lines. MRI (Magnetic Resonance Imaging) is a radiology imaging technique that avoids radiation exposure to the patient). Solving a system of equations allows to compute the actual densities and so to do the magic of "seeing inside the body".

We build the augmented matrix $[A|b]$ and row reduce. First remove the sum of the first three rows from the 4th, then change the sign of the 4'th column:

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \\ 1 & 1 & 1 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & -1 & -1 & -6 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we can read of the solutions. We see that v and w can be chosen freely. They are free variables. We write $v = r$ and $w = s$. Then just solve for the variables:

$$\begin{aligned} x &= -6 + r + s \\ y &= 5 - r \\ z &= 9 - s \\ u &= 9 - r - s \\ v &= r \\ w &= s \end{aligned}$$

Homework

Problem 2.1: For a **polyhedron** with v vertices, e edges and f triangular faces Euler proved his famous formula $v - e + f = 2$. An other relation $3f = 2e$ called a Dehn-Sommerville relation holds because each face meets 3 edges and each edge meets 2 faces. Assume the number the number f of triangles is 288. Write down a system of equations for the unknowns v, e, f in matrix form $Ax = b$, then solve it to find v and e.

Problem 2.2: Row reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$.

Problem 2.3: a) In the "Nine Chapters on Arithmetic", the following system of equations appeared $3x+2y+z = 39, 2x+3y+z = 34, x+2y+3z = 26$. Solve it using row reduction by writing down an augmented matrix and row reduce.

Problem 2.4: a) Which of the following matrices are in row reduced echelon form?
$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
b) Two $n \times m$ matrices in reduced row-echelon form are called **of the same type** if they contain the same number of leading l's in the same positions. For example, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

Problem 2.5: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Compare $\text{rref}(A^T)$ with $(\text{rref}(A))^T$. Is it true that the transpose of a row reduced matrix is a row reduced matrix?