Unit 4: Cross product

Lecture

4.1. The three dimensional space $\mathbb{R}^3$ is special. It is not only the only Euclidean space in which the Kepler problem is stable [1], it also features a **cross product** $v \times w$ which is in the same space. Such a product can be defined in $\mathbb{R}^n$ but it produces a vector in $\mathbb{R}^{n(n-1)/2}$. It happens that for $n = 3$ that the result is again in $\mathbb{R}^3$. The problem of "multiplying triplets" has been pondered by William Hamilton in the first half of the 19th century and is related to the fascinating story of **quaternions**. The discovery of quaternions was simultaneously the birth place of the dot and cross product.

4.2. The **cross product** of two vectors $v = [v_1, v_2, v_3]^T$ and $w = [w_1, w_2, w_3]^T$ is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} .$$

Take the dot product with v or w to see that $v \times w$ is perpendicular to both v and w. Obvious is also $v \times w = -w \times v$. The product is handy for constructions in $\mathbb{R}^3$. The vectors $v, w, v \times w$ are oriented like the first three fingers on the **right hand**: if v is the thumb, w is the pointing finger, then $v \times w$ is the middle finger. Let $v \cdot w = |v||w|\cos(\alpha)$:

Theorem: $|v \times w| = |v||w|\sin(\alpha)$ and $v \cdot (v \times w) = w \cdot (v \times w) = 0$.

Proof. We will verify in class by brute force the **Lagrange's identity** $|v \times w|^2 = |v|^2|w|^2 - (v \cdot w)^2$ which is also called **Cauchy-Binet** formula. Now use $|v \cdot w| = |v||w|\cos(\alpha)$ to get the result with $\cos^2(\alpha) + \sin^2(\alpha) = 1$. □

4.3. Given a triangle with side lengths a, b, c and angles α, β, γ, where α is opposite to a etc. We have the following sin-**formula**

Corollary: $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$.

Proof. We can use the theorem and express the area of the triangle as $ab\sin(\gamma)$ or $bc\sin(\alpha)$ or $ac\sin(\beta)$. By equating these three quantities and dividing out the common factor, we get the sin-formula. □

[1] by a theorem of Joseph Bertrand of 1873 and work of Sundman-von Zeipel

4.4. This is useful in applications as to define the area of the parallelogram as $|v \times w|$. That this is justified can be seen in two dimensions and:

Corollary: $|v \times w|$ is the **parallelogram area** spanned by v and w.

Proof. Use the formula $|v \times w| = |v||w|\sin(\alpha)$ and note that $|w|\sin(\alpha)$ is the height of the parallelogram spanned by v and w. The base length is $|v|$. □

4.5. The scalar $u \cdot (v \times w)$ is called the **triple scalar product** of u, v, w. Its **sign** defines an **orientation** of the three vectors. It is also the **determinant** of the matrix

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} .$$

The absolute value of $u \cdot v \times w$ defines the **volume of the parallelepiped** spanned by u, v and w. Without the absolute value, we also speak of **signed volume**.

4.6. Side remark: In higher dimensions, the cross product is called **exterior product**. One uses $\wedge$ rather than $\times$ which is used in three dimensions. If $I = (i, j)$ is a choice of two elements in $\{1, 2, \dots, n\}$ and v, w are two vectors in $\mathbb{R}^n$, then $(v \wedge w)_I = v_i w_j - v_j w_i$. The formula $|v \wedge w| = |v||w|\sin(\alpha)$ still holds and the proof is the same. We only need again to verify the **Cauchy-Binet** formula $|v|^2|w|^2 - (v \cdot w)^2 = |v \wedge w|^2$. But this is better done using matrices. If A is the matrix which contains v, w as columns, then $\det(A^T A) = \sum_P \det(A_P)^2$, where the sum on the right is over all 2×2 submatrices A_P of A. The expression $\det(A_P)$ is called a **minor**. Cauchy-Binet formula is super cool [2]. By the way, if we have k vectors and build $A \in M(n, k)$, a matrix which has these vectors as columns. Now, $\det(A^T A)$ is the volume of the parallelepiped spanned by these vectors. And Cauchy-Binet writes this as a sum of squares of k-dimensional volumes of projections which is in some sense a generalization of Pythagoras.

Examples

4.7. What is the area of the triangle $A = (1, 1, 1)$, $B = (3, 5, 2)$ and $C = (2, 0, 3)$? We find the cross product between the vector $[2, 4, 1]^T$ going from A to B and the vector $[1, -1, 2]^T$ going from A to C. The cross product is

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix} .$$

Its length is $3\sqrt{14}$. The area of the triangle is half of it: $3\sqrt{14}/2$.

4.8. Find the volume of the parallelepiped for which one of the vertices is $(0, 0, 0)$ and the other neighbors are A, B, C from before? We find the signed volume

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot (\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ -5 \\ -8 \end{bmatrix} = -1 .$$

and take the absolute value. A negative number indicates that OA, OB, OC is left handed.

[2] O. Knill, Cauchy Binet for pseudo-determinants, Lin. Alg. and its Applications 459 (2014) 522-547

Illustrations

Figure 1. The just newly released Swiss 200 Frank bill shows the **right hand rule**: thumb = v, pointing finger = w, then $v \times w$ is the middle finger. Source: Swiss National Bank, issued August 22, 2018.

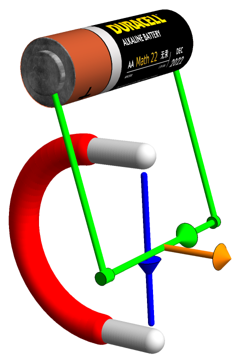

Figure 2. The **Lorentz force** F is a vector $F = qv \times B$ determined by the velocity v of a charged particle with charge q moving in a magnetic field B.

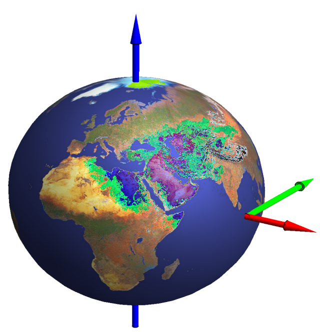

Figure 3. Given a particle of mass m at position r moving with the velocity r' then $L = mr \times r'$ is the **angular momentum**.

Homework

Problem 4.1: Find a vector w perpendicular to the vectors $u = [1,1,1]^T$ and $v = [3,4,5]^T$. Then use this result to find a vector x perpendicular to both v and w.

Problem 4.2: A **3D scanner** is used to build a 3D model of a face. It detects a triangle which has its vertices at $P = (0,1,1), Q = (1,1,0)$ and $R = (1,2,3)$. Find the area of that triangle as well as a vector perpendicular to the triangle. (*)

Problem 4.3: a) Find the volume of the parallelepiped which has the vertices $O = (0,0,0), P = (2,3,1), Q = (4,3,1), R = (6,6,2)$. $A = (1,1,1), B = (3,4,2), C = (5,4,2), D = (7,7,3)$.

Problem 4.4: Investigate which of the following formulas are always true for all vectors u,v,w,x,y. If it is true, either explain, cite a source (i.e. on the web), or a by hand or computer algebra verification. If it is not true, find a counter example.
a) $u \times (v \times w) = (u \times v) \times w$
b) $u \cdot (v \times w) = v \cdot (w \times u)$
c) $u \times (v + w) = u \times v + u \times w$
d) $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$.
e) $(u \times v) \cdot (x \times y) = (u \cdot x)(v \cdot y) - (u \cdot y)(v \cdot x)$.

Problem 4.5: Given two vectors $p = [a,b,c]^T$ and $q = [u,v,w]^T$, build the matrices $P = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $Q = \begin{bmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{bmatrix}$ Compare $p \times q$ and $QP - PQ$. Describe what you see.

(*) The STL format which is used for 3D printing, has an extremely simple form. It consists of entries like

```
facet normal 0.15 -0.97 -0.20
outer loop
vertex -1.6996 -0.5597 -2.8360
vertex -1.8259 -0.5793 -2.8374
vertex -1.7232 -0.5399 -2.9509
endloop
endfacet
```

The first line gives the normal vector, then there is a loop with three vertices giving the triangle. There is obviously some redundancy as one could get the normal vector from the points using the cross product. But there is purpose: the redundant information makes working with the data structure faster, second, one can also look at situations, where the normal vector is not perpendicular to the surface, one can change the way how the is “shaded”, like how light is reflected at the surface. Third, redundancy is always good to catch errors. Our genetic information in the DNA is stored in a highly redundant way. This allows error correction.