LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 7: Curves

Lecture

7.1. Given n continuous functions $x_j(t)$ of one variable t, we can look at the vector-valued function $r(t) = [x_1(t), \dots, x_n(t)]^T$. We call it a **parametrized curve**. An example is $r(t) = [3+2t, 4+6t]$ which is a line through the point $(3,4)$ and containing the vector $[2,6]$. [1] If t is in the **parameter interval** $a \leq t \leq b$, then the image of r is $\{r(t) \mid a \leq t \leq b\}$, which defines a **curve** in $\mathbb{R}^n$. The curve **starts** at the point $r(a)$ and **ends** at the point $r(b)$. An other important example is the **circle** $r(t) = [\cos(t), \sin(t)]$, where t is in the interval $[0, 2\pi]$. Its image is a circle in the plane $\mathbb{R}^2$. The parametrization $r(t)$ contains more information than the curve itself: the parabolic curve $r(t) = [t, t^2]$ defined on $t \in [-1,1]$ for example is the same as the curve $r(t) = [t^3, t^6]$ for $\in [-1,1]$, but in the second parametrization, the curve is traveled with different speed. Curves in $\mathbb{R}^3$ can be admired in our physical space like $r(t) = [x(t), y(t), z(t)] = [t\cos(t), t\sin(t), t]$ which is a spiral. You can see that this particular curve is contained in the cone $x^2 + y^2 = z^2$.

7.2. If the functions $t \to x_j(t)$ are differentiable, we can form the derivative $r'(t) = [x_1'(t), \dots, x_n'(t)]$. While this technically is again a curve, we think of $r'(t)$ as a vector attached to the point $r(t)$ and say that $r'(t)$ is **tangent** to $r(t)$. The length $|r'(t)|$ of the velocity is called the **speed** of r. If also higher derivatives of the functions $x_j(t)$ exist, we can form the second derivative $r''(t)$ called the **acceleration** or third derivative $r'''(t) = r^{(3)}(t)$ called the **jerk**. Then come **snap** $r^{(4)}(t)$, **crackle** $r^{(5)}(t)$ and **pop** $r^{(6)}(t)$ and the **Harvard** $r^{(7)}(t)$ introduced in the fall of 2016 in a multi-variable exam.

7.3. Given the first derivative function $r'(t)$ as well as the initial point $r(0)$, we can get back the function $r(t)$ thanks to the **fundamental theorem of calculus**. Because of **Newton's law** which tells that a mass point of mass m subject to a force field F depending on position and velocity satisfies the **Newtonian differential equation** $mr''(t) = F(r(t), r'(t))$, the following result is important:

Theorem: $r(t)$ is uniquely determined from $r''(t)$ and $r(0)$ and $r'(0)$.

Proof. In each coordinate we get $x_k'(t) = \int_0^t x_k''(s)\, ds + x_k'(0)$ and $x_k(t) = \int_0^t x_k'(s)\, ds + x_k(0)$. We have just applied twice the **fundamental theorem of calculus**. $\square$

[1] To reduce clutter, we write row vectors $[2,6]$ rather than column vectors

A special case is if $r''(t)$ is constant. A special case is the **free fall situation**. The coordinate functions are then quadratic. Assume $r''(t) = [0, 0, -10]$, and $r'(0) = [0, 0, 0]$ and $r(0) = [0, 0, 20]$, then $r(t) = [0, 0, 20 - 5t^2]$. If you jump from 20 meters into a pool, you need $t = 2$ seconds to hit the water.

7.4. Given a curve $r(t)$ for which the velocity $r'(t)$ is never zero, we can form the **unit tangent vector** $T(t) = r'(t)/|r'(t)|$. If $T'(t)$ is never zero, we can then form $N(t) = T'(t)/|T'(t)|$, the **normal vector**. The vector $B = T \times N$ is called the **binormal vector**. The scalar $|T'(t)|/|r'(t)|$ is called the **curvature** of the curve.

Theorem: In $\mathbb{R}^3$, we have $K = |T'|/|r'| = |r' \times r''|/|r'|^3$.

Proof. We will do this computation in class. □

7.5. Even if $r(t)$ is perfectly smooth, the curvature can become infinite. Lets look at the example $r(t) = [t^2, t^3, 0]$. Then $r'(t) = [2t, 3t^2, 0]$ and $r''(t) = [2, 6t, 0]$ and $r'(t) \times r''(t) = [0, 0, 6t^2]$. The curvature is $(6/t)(4 + 9t^2)^{-3/2}$ which has a singularity at $t = 0$.

7.6. Even when $r(t)$ is perfectly smooth and never zero, the normal vector can depend in a discontinuous way on t. Example: $r(t) = [t, t^3/3]$. Now $r'[t] = [1, t^2]$ and $T(t) = [0, t^2]/\sqrt{1 + t^4}$. We see that $T'(t)$ takes different signs in the second coordinate. After normalization we have $\lim_{t \to 0, t>0} N(t) = [0, 1]$ and $\lim_{t \to 0, t<0} N(t) = [0, -1]$. At the **inflection point** of the graph of the cube function, the concavity has changed from concave down to concave up. This has changed the direction of the normal vector N.

7.7. Side remark. We have looked at parametrized vectors only. If the entries $A_{ij}(t)$ of a matrix depend on times we have a matrix valued curve $A(t)$. This appears in differential equations, in quantum mechanics (operators moving in time) or - most importantly - in moving pictures! A movie is just a matrix valued curve.

7.8. Side remark. A planar curve $r(t) = [x(t), y(t)]^T$ in the plane defined on $t \in [0, 2\pi]$ is called a **simple closed curve** if $r(0) = r(2\pi)$ and there are no values $0 \leq s \neq t < 2\pi$ for which $r(t) = r(s)$. For a smooth curve, meaning that the first two derivatives exist, we can look at the polar angle $\alpha(t)$ of the vector $r'(t)$. Define the **signed curvature** of the curve as $\kappa(t) = \alpha'(t)/|r'(t)|$. We have $|\kappa(t)| = K(t)$. The **Hopf Umlaufsatz** tells $\int_0^{2\pi} \kappa(t) \; dt = 2\pi$. In the case of the circle for example, $\kappa(t) = 1$.

7.9. Side remark. We can verify that any curve $r(t)$ parametrized on $[a, b]$ such that $r'(t) \neq 0$ for all $t \in [a, b]$ can be parametrized as $R(t)$ on $[a, b]$ such that $|R'(t)| = 1$ for all t. Proof: we look for a monotone function $s(t)$ such that the derivative of $r(s(t))$ has length 1. This means we want $|r'(s(t))| s'(t) = 1$. In other words, look for a function $s(t)$ such that $s'(t) = 1/|r'(s(t))| = F(s(t))$ and $s(a) = 0$. This is what we call a differential equation. There is a general existence theorem for differential equations (proven later) which assures that there exists a unique solution $s(t)$. End of proof. The result is very intuitive. You can drive from $r(a)$ to $r(b)$ along the curve traced by $r(t)$ by just keeping the speed 1. This gives your your new parametrization. Your new time interval will be $[0, L]$ where L is the arc length (the length of your trip). We will come to arc length computation in the next lesson.

7.10. Side remark. Continuous curves can be complicated: If you look at the pollen particle in a microscope, it moves erratically on a curve which is nowhere differentiable as it is constantly bombarded with air molecules which bounce it around. This is **Brownian motion**. There are also **Peano curves** or **Hilbert curves** $[0,1] \to [0,1]^2$ or space filling Hilbert curves $r(t) : [0,1] \to Q = [0,1]^3$ which cover every point of the **cube** Q. These curves define a continuous bijection from $[0,1]$ to $[0,1]^3$. (The inverse is not continuous. Still, the construction shows that there are the same number of points in $[0,1]$ than in $[0,1]^3$).

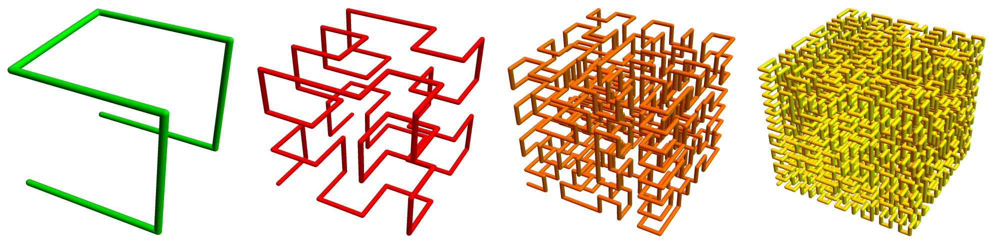

FIGURE 1. The four first stages in the construction of a space filling curve.

EXAMPLES

7.11. Assuming the **Newton equations** $mr''(t) = F(t)$, find the path $r(t)$ of a body of mass $m = 1/2$ subject to a force $F(t) = [\sin(t), \cos(t), -10]$ with $r(0) = [3,4,5]$ and $r'(0) = [1,2,7]$. Solution: we have $r''(t) = [2\sin(t), 2\cos(t), -20]$. Integration gives $r'(t) = [-2\cos(t), 2\sin(t), -20t] + [c_1, c_2, c_3]$. Fixing the constants gives $r'(t) = [3 - 2\cos(t), 2 + 2\sin(t), 7 - 20t]$. A second integration gives $r(t) = [3t - 2\sin(t), 2t - 2\cos(t), 7t - 10t^2] + [c_1, c_2, c_3]$ with other constants $C = [c_1, c_2, c_3]$. Comparing $r(0) = [0, -2, 0] + [c_1, c_2, c_3] = [3,4,5]$ gives $r(t) = [3+3t-2\sin(t), 6+2t-2\cos(t), 5+7t-10t^2]$.

7.12. Let $r(t) = [L\cos(t), L\sin(t), 0]$. Then $r'(t) = [-L\sin(t), L\cos(t), 0]$ and $r''(t) = [-L\cos(t), -L\cos(t), 0]$ and $r'(t) \times r''(t) = [0, 0, L^2]$ and $|r'(t)| = L$ so that $|r'(t) \times r''(t)|/|r'(t)|^3 = 1/L$. A circle of radius L has curvature $1/L$!

7.13. A closed simple curve C in $\mathbb{R}^3$ is a **knot**. For any positive integer n, m we can look at the **torus knot** $r(t) = [(3 + \cos(mt))\cos(nt), (3 + \cos(mt))\sin(nt), \sin(mt)]$. The **total curvature** of a knot is defined as $\int_0^{2\pi} K(t)\, dt$. See Figure 2. [2]

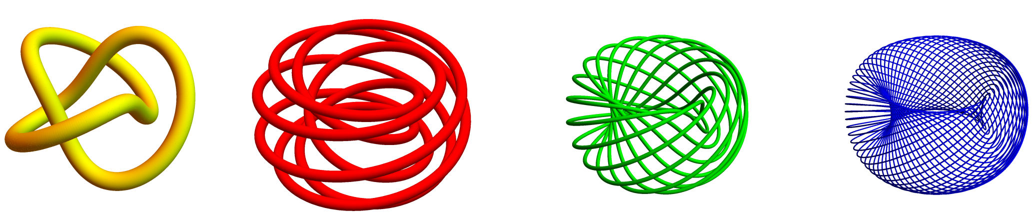

FIGURE 2. Torus knots $T(2,3), T(7,3), T(12,13)$ and $T(30,43)$. Their total curvatures are $38.6, 245.6, 487.2, 2167.3$.

[2] A general theorem of Fay and Milnor assures that a knot of total curvature $\leq 4\pi$ is trivial.

Homework

Problem 7.1: A stone of mass $m = 0.1$ in the **Pandora Halleluya mountains** is exposed to the force $F(t) = [\log(e+t), e^{t/100}, \sin(t)]$. It is initially at $r(0) = [0, 0, 100]$ and has zero initial velocity $r'(0) = [0, 0, 0]$. Where is it at $t = 10$? In this course, we always write $\log(t) = \ln(t)$.

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Problem 7.2: We want to produce a logo for a new company and experiment. Draw the curve $r(t) = [\cos(t), \sin(t)] + [\cos(5t), \sin(7t)]/4 + [\cos(13t), \sin(9t)]/4$ and find the velocity, acceleration, and curvature at $t = 0$.

Problem 7.3: Parametrize the curve $r(t)$ obtained by intersecting the cylinder $x^2/9 + y^2/4 = 1$ with the plane $z = x + 5y$.

Problem 7.4: Verify that the **torus knot** $r(t) = [x(t), y(t), z(t)] = [(2+\cos(mt))\cos(nt), (2+\cos(mt))\sin(nt), \sin(mt)]$ lives on the torus $(3+x^2+y^2+z^2)^2 - 16(x^2+y^2) = 0$.

Problem 7.5: In the lecture on surfaces, we have sliced some bagels. Let us assume that the doughnut is given by $(x^2+y^2+z^2+16)^2 - 100(x^2+y^2) = 0$. Verify that if we intersect this torus with the plane $3x = 4z$, then we get the **Villarceau circles** $r(t) = [4\cos(t), 3+5\sin(t), 3\cos(t)]$ as well as the circle $r(t) = [4\cos(t), -3+5\sin(t), 3\cos(t)]$.

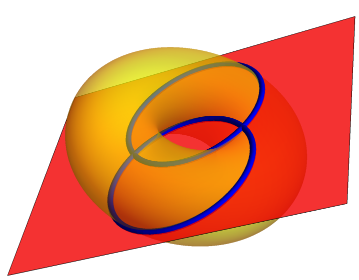

Figure 3. Villarceau circles.

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[3] The notation ln appears only in calculus books. Mathematicians use log.