LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 9: Intuition

Seminar

9.1. It is important in mathematics to gain intuition about objects, definitions and theorems and proofs. The fact that this is not easy can be illustrated by showing that intuition can mislead us. We can state "false theorems" which we would believe to be true but which are false. We start with the notion of "continuity" for which an intuitive definition tells: we can "draw the graph of a continuous function without having to lift the pen". Of course, we can not work with this definition to prove theorems.

9.2. Starting with Cauchy and pushed heavily by Weierstrass, continuity is defined precisely using the infamous $\epsilon - \delta$ definition: f is continuous at x, if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq \epsilon$. Using more fancy mathematical quantifier notation $\forall$ (for all) and $\exists$ (exists) and $\Rightarrow$ (implies) and ϵ (is element of) you can impress your friends (and annoy readers and graders) by writing

$$\boxed{\forall \epsilon > 0 \exists \delta > 0 \forall y \in [a, b], |x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon\ .}$$

The fact that his definition is not intuitive at all and that most students just learn this "epsilontic" by **intimidation** is illustrated by the following variation by Ed Nelson [1] We make it our first exercise:

Problem A: What does the following statement mean?

$$\boxed{\forall \delta > 0 \exists \epsilon > 0 \forall y \in [a, b], |x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon\ .}$$

9.3. In the first lecture we have seen how a polygonal approximation of a curve allows to compute the arc length of a curve. Here is a first "anti-theorem". Your task is to figure out what is wrong.

[1] E. Nelson, Internal set theory: A new approach to nonstandard analysis, 1977

9.4. We compute the **circumference of a circle** by a polygonal approximation. The following statement uses the intuition that if a polygon is close to a curve, then its length is close to the curve:

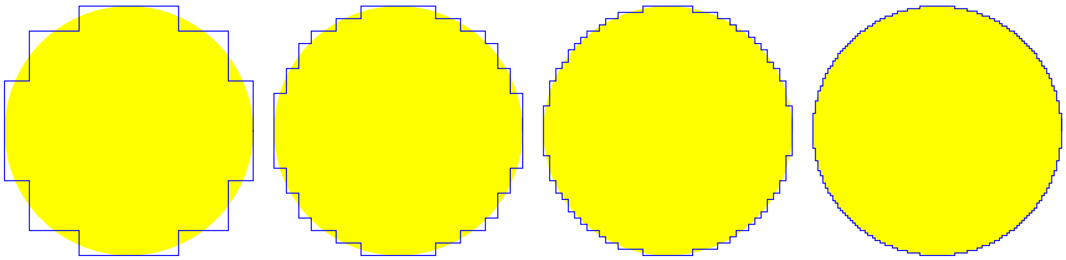

FIGURE 1. The circumference of a circle is 8.

9.5. This leads to the following anti-theorem: [2] A continuous planar curve is a function $t \to r(t) = [x(t), y(t)]$, where both functions $x(t), y(t)$ are continuous functions.

False Theorem: The circumference of the unit circle is 8.

Problem B: What is wrong with the argumentation?

9.6. We could also think that the arc length of a continuous curve is finite.

False Theorem: The arc length of a continuous curve is finite.

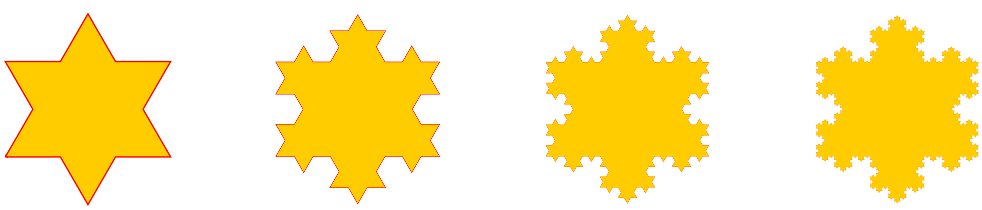

FIGURE 2. The first 4 approximations of the Koch snowflake.

Problem C: Find a formula for the length of the k'th Koch curve approximation if initially, the triangle has side length 1

[2] Again thanks to Jun Hou Fung for suggestion

9.7. If a curve $t \to r(t) = [x(t), y(t)]$ has the property that $x(t)$ and $y(t)$ stay bounded and have no jump discontinuities, we would think that the curve is continuous.

False Theorem: A bounded curve without jumps is continuous.

9.8. A counter example is the **devil comb** $r(t) = [t, \sin(1/t)]$ for $t \in [0,1]$. it does not have a jump discontinuity and it is bounded. The function is not defined at $t = 0$ but we can define $r(0) = [0,0]$ to make it defined anywhere on $[0,1]$.

Problem D: Why is this function $r(t)$ not continuous at $t = 0$?

9.9. Finally, we could think:

False Theorem: A continuous function is differentiable at some point.

9.10. A counter example was given by Weierstrass. It is called the Weierstrass function. G.H. Hardy proved in 1916 that the function

$$f(x) = \sum_{n=1}^{\infty} a^{-n} \cos(a^n x)$$

does not have any point of differentiability if $a > 1$.

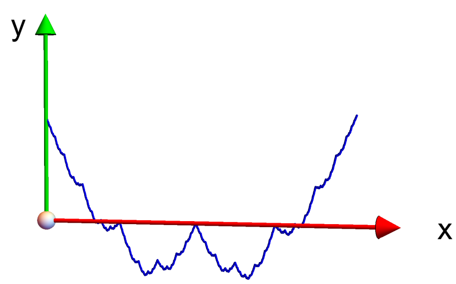

FIGURE 3. The Weierstrass function for $a = 2$, displayed on $[0, \pi]$.

Problem E: Show that $f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos(2^n x) \in [-1, 1]$.

Homework

Exercises A-E are done in the seminar. This homework is due on Tuesday:

Problem 9.1 Prove that there was a time in your life when the length of your largest tooth in millimeters was your height in meters.

Problem 9.2 Use the intermediate value theorem to prove the mean value theorem: if f is continuously differentiable and $f(0) = f(1) = 0$, then there exists a point in $(0, 1)$ with $f'(x) = 0$.

Problem 9.3 Look up, formulate and understand the proof of the "Wobbly table theorem". This theorem appears to have been found in 2008 by David Richeson. You find an exposition in some of Harvard Math 1a handouts.

Problem 9.4 We can draw curves in 4 dimensions by assigning to each point a **Hue value**, which is a color parametrized by $[0, 1]$. Given a curve $(x(t), y(t), z(t))$ and a color value $c(t)$ define the curve $r(t) = [x(t), y(t), z(t), c(t)]$ in four dimensional space. Use this idea to argue why there are no non-trivial knots in four dimensions.

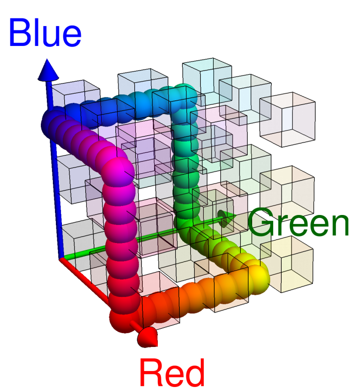

Figure 4. The "Hue curve" in color space.

Problem 9.5 What does your intuition say? We will come back to this later when we look at surface area. Given a nice smooth surface S like a paraboloid which is triangulated with triangles of size ϵ. If S_n is the polygonal approximation. Does the surface area $|S_n|$ of the polyhedron and the surface area of the surface S satisfies $|S_n| \to |S|$?