LINEAR ALGEBRA AND VECTOR ANALYSIS

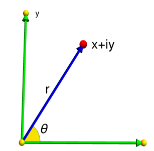
MATH 22A

Unit 10: Coordinates

Lecture

10.1. It was René Descartes who in 1637 introduced coordinates and brought algebra close to geometry. [1] The **Cartesian coordinates** (x, y) in $\mathbb{R}^2$ can be replaced by other coordinate systems like **polar coordinates** (r, θ), where $r = \sqrt{x^2 + y^2} \geq 0$ is the **radial distance** to the $(0, 0)$ and $\theta \in [0, 2\pi)$ is the **polar angle** made with the positive x-axis. Since θ is in the interval $[0, 2\pi)$, it is best described in the complex notation $\theta = \arg(x + iy)$. The conversion from the (r, θ) coordinates to the (x, y)-coordinates is

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$



The radius is $\sqrt{x^2 + y^2}$, where if non-zero, we always take the positive root. The angle formula $\arctan(y/x)$ only holds if x and y are both positive. The angle θ is not uniquely defined at the origin $(0, 0)$, most software just assumes $\arg(0) = 0$.

10.2. We can write a vector in $\mathbb{R}^2$ also in the form of a **complex number** $z = x + iy \in \mathbb{C}$ with some symbol i. This is not only notational convenience. Complex numbers can be added and multiplied like other numbers and while $\mathbb{R}^2 = \mathbb{C}$, the later has a **multiplicative structure**. In order to fix that structure, one only needs to specify that $i^2 = -1$. This gives $(a + ib)(c + id) = ac - bd + i(ad + bc)$. An important observation of Euler is a link between the exponential and trigonometric functions:

Theorem: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

10.3. The proof is to write the series definition on both sides. First recall the definitions of $e^x = 1 + x + x^2/2! + x^3/3! + ...$ If we plug in $x = i\theta$ we get $e^{i\theta} = 1 + i\theta - \theta^2/2! - i\theta^3/3! + \theta^4/4!...$ But this is $(1 - \theta^2/2 + \theta^4/4!...) + i(\theta - \theta^3/3! + \theta^5/5! - ...)$ which is $\cos(\theta) + i\sin(\theta)$. QED. If you prefer not to see the functions $\exp, \sin, \cos$ being **defined** as series, you can see them as **Taylor series** $f(x) = f(0) + f'(0)x + f''(0)/2!x^2 + ... = \sum_{k=0}^{\infty}(f^{(k)}(0)/k!)x^k$. By differentiating the functions at 0, we see then the connection.

[1] Descartes: La Géometrie, 1637 (1 year after the foundation of Harvard college)

10.4. This implies for $\theta = \pi$ the magical formula

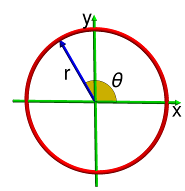
Theorem: $e^{i\pi} + 1 = 0$

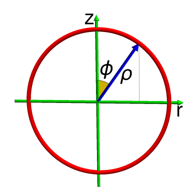
This formula is often voted the **"nicest formula in math"**. [2] It combines "analysis" in the form e, "geometry" in the form of π, "algebra" in the form of i, the additive unit 0 and the multiplicative unit 1. The Euler formula also allows to define the logarithm of any complex number as $\log(z) = \log(|z|) + i\arg(z) = \log(r) + i\theta$. We see now that going from (x, y) to $(\log(r), \theta)$ is a very natural transformation from $\mathbb{C} \setminus 0$ to $\mathbb{C}$. The exponential function $\exp : z \to e^z$ is a map from $\mathbb{C} \to \mathbb{C} \setminus 0$. It transforms the additive structure on $\mathbb{C}$ to the multiplicative structure because $\exp(z + w) = \exp(z)\exp(w)$.

10.5. In three dimensions, we can look at **cylindrical coordinates** (r, θ, z). It is just the polar coordinates in the first two coordinates. A cylinder of radius 2 for example is given as $r = 2$. The torus $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$ can be written as $3 + r^2 + z^2 = 4r$ or more intuitively as $(r - 2)^2 + z^2 = 1$, a circle in the $r - z$ plane.

10.6. The **spherical coordinates** (ρ, θ, ϕ), where $\rho = \sqrt{x^2 + y^2 + z^2}$. The angle θ is the polar angle as in cylindrical coordinates and ϕ is the angle between the point (x, y, z) and the z-axis. We have $\cos(\phi) = [x, y, z] \cdot [0, 0, 1]/|[x, y, z]| = z/\rho$ and $\sin(\phi) = |[x, y, z] \times [0, 0, 1]|/|[x, y, z]| = r/\rho$ so that $z = \rho\cos(\phi)$ and $r = \rho\sin(\phi)$ and therefore

$$\begin{aligned} x &= \rho\sin(\phi)\cos(\theta) \\ y &= \rho\sin(\phi)\sin(\theta) \\ z &= \rho\cos(\phi) \end{aligned}$$





where $0 \le \theta < 2\pi, 0 \le \phi \le \pi$ and $\rho \ge 0$.

10.7. A **coordinate change** $x \to f(x)$ in the plane can be seen as a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. A point (x_1, x_2) is mapped into (f_1, f_2). We write ∂_{x_k} for the **partial derivative** with respect to the variable x_k. For example $\partial_{x_1}(x_1^2 x_2 + 3x_1 x_2^3) = 2x_1 x_2 + 3x_2^3$.

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}, \quad df\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) \end{bmatrix},$$

where df is a matrix called the **Jacobian matrix**. The determinant is called the **distortion factor** at $x = (x_1, x_2)$.

10.8. For polar coordinates, we get

$$f\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}, df\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}.$$

Its distortion factor is r. We will use this when integrating in polar coordinates.

10.9. If $f(z) = z^2 + c$ with $c = a + ib$, $z = x + iy$ is written as $f(x, y) = (x^2 - y^2 + a, 2xy + b)$, then df is a 2×2 **rotation dilation matrix** which corresponds to the complex number $f'(z) = 2z$. The algebra $\mathbb{C}$ is the same as the algebra of rotation-dilation matrices.

[2] D. Wells, Which is the most beautiful?, Mathematical Intelligencer, 1988

10.10. A **coordinate change** $x \to f(x)$ in space is a map $f : \mathbb{R}^3 \to \mathbb{R}^3$. We compute

$$f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}, df\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) & \partial_{x_3} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) & \partial_{x_3} f_2(x) \\ \partial_{x_1} f_3(x) & \partial_{x_2} f_3(x) & \partial_{x_3} f_3(x) \end{bmatrix}.$$

We wrote $x = (x_1, x_2, x_3)$. Its determinant $\det(dT)(x)$ is a volume distortion factor.

10.11. For spherical coordinates, we have

$$f\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \rho\sin(\phi)\cos(\theta) \\ \rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi) \end{bmatrix}, df\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{bmatrix}.$$

The distortion factor is $\det(df(\rho, \phi, \theta)) = \rho^2 \sin(\phi)$.

Examples

10.12. The point $(x, y) = (-1, 1)$ corresponds to the complex number $z = -1 + i$. It has the polar coordinates $(r, \theta) = (\sqrt{2}, 3\pi/4)$. As we have $z = re^{i\theta}$, we check $z^2 = (-1+i)(-1+i) = -2i$ which agrees with $(re^{i\theta})^2 = r^2 e^{2i\theta} = 2e^{6\pi i/4}$.

10.13. a) $(x, y, z) = (1, 1, -\sqrt{2})$ corresponds to spherical coordinates $(\rho, \phi, \theta) = (2, 3\pi/4, \pi/4)$.
b) The point given in spherical coordinates as $(\rho, \phi, \theta) = (3, 0, \pi/2)$ is the point $(0, 3, 0)$.

10.14. a) The set of points with $r = 1$ in $\mathbb{R}^2$ form a circle.
b) The set of points with $\rho = 1$ in $\mathbb{R}^3$ form a sphere.
c) The set of points with spherical coordinates $\phi = 0$ are points on the positive z-axis.
d)The set of points with spherical coordinates $\theta = 0$ form a half plane in the yz-plane.
e) The set of points with $\rho = \cos(\phi)$ form a sphere. Indeed, by multiplying both sides with ρ, we get $\rho^2 = \rho\cos(\phi)$ which means $x^2 + y^2 + z^2 = z$, which is after a completion of the square equal to $x^2 + y^2 + (z - 1/2)^2 = 1/4$.

10.15. For $A \in M(n, n)$, $f(x) = Ax + b$ has $df = A$ and distortion factor $\det(A)$.

10.16. Find the Jacobian matrix and distortion factor of the map $f(x_1, x_2) = (x_1^3 + x_2, x_2^2 - \sin(x_1))$. Answer: Write both the transformation and the Jacobian:

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^3 + x_2 \\ x_2^2 - \sin(x_1) \end{bmatrix}, \; df\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1^2 & 1 \\ -\cos(x_1) & 2x_2 \end{bmatrix}.$$

The Jacobian matrix is $\det(df(x)) = 6x_1^2 x_2 + \cos(x_1)$.

Illustrations

10.17. Let $T : \mathbb{C} \to \mathbb{C}$ be defined as $z \to z^2 + c$ where $z = x + iy$. The set of all $c = a + ib$ for which the iterates $T^n(0)$ stay bounded is the **Mandelbrot set** M. For $c = -1$ we get $T(0) = -1, T^2(0) = T(-1) = 0$ so that $T^n(z)$ is either 0 or -1. The point $c = -1$ is in M. The point $c = 1$ gives $T(0) = 1, T^2(0) = 1^2 = 1 = 2, T^3(0) = 2^2 + 1 = 5$. Induction shows that $T^n(0)$ does not converge. The point $c = 1$ is not in M.

10.18. If T is the transformation in $\mathbb{R}^3$ which is in spherical coordinates given by $T(x) = x^2 + c$, where x^2 has spherical coordinates $(\rho^2, 2\phi, 2\theta)$ if x has (ρ, ϕ, θ). It turns out that $T(x) = x^8 + c$ gives a nice analogue of the Mandelbrot set, the **Mandelbulb**.

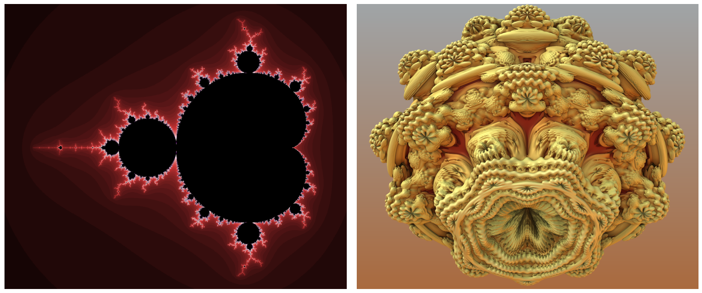


FIGURE 1. The **Mandelbrot set** $M = \{c \in \mathbb{C} | T(z) = z^2 + c$ has bounded $T^n(0)\}$. There is a similar construction in space $\mathbb{R}^3$ which uses spherical coordinates. This leads to the **Mandelbulb set** $B = \{c \in \mathbb{R}^3 \mid T(x) = x^8 + c$ has bounded $T^n(0)\ \}$, where x^8 has spherical coordinates $(\rho^8, 8\phi, 8\theta)$ if x has spherical coordinates (ρ, ϕ, θ).

Homework

Problem 10.1: a) Find the polar coordinates of $(x, y) = (1, \sqrt{3})$.
b) Which point has the polar coordinates $(r, \theta) = (3, 4)$?
c) Find the spherical coordinates of the point $(x, y, z) = (1, 1, 1)$.
d) Which point has the spherical coordinates $(\rho, \theta, \phi) = (3, \pi/2, \pi/3)$?

Problem 10.2: a) Compute $T_c^n(0)$ for $c = (1 + i)$ for $n = 1, 2, 3, 4$. Is $1 + i$ in the Mandelbrot set?
b) What is the "eye for an eye" number i^i? (You can use $z^w = e^{w \log(z)}$).

Problem 10.3: a) Which surface is described as $r = z$?
b) Describe the hyperbola $x^2 - y^2 = 1$ in polar coordinates.
c) Which surface is described as $\rho \sin(\phi) = \rho$?
d) Describe the hyperboloid $x^2 + y^2 - z^2 = 1$ in spherical coordinates.

Problem 10.4: a) Compute the Jacobian matrix and distortion factor of the coordinate change $T(x, y) = (2x + \sin(x) - y, x)$ (**Chirikov map**).
b) Compute for fixed a the Jacobian matrix and distortion factor of the toral coordinates $f(q, \theta, \phi) = ((a + q\cos(\phi))\cos(\theta), (a + q\cos(\phi))\sin(\theta), q\sin(\phi))$. The θ and ϕ coordinates are angles, the q coordinate is the distance to the center circle of the torus.

Problem 10.5: a) Prove by induction that the Mandelbrot set M is contained in the set $|c| \leq 2$.
b) Prove by induction that the Mandelbulb set B is contained in the set $|c| \leq 2$.