Unit 11: Parametrization

Lecture

11.1. A map $r : \mathbb{R}^m \to \mathbb{R}^n$ is called a **parametrization**. We have seen maps r from $\mathbb{R}$ to $\mathbb{R}^n$, which were **curves**. Then we have seen maps $f : \mathbb{R}^n \to \mathbb{R}^n$ which were **coordinate changes**. In each case we defined the **Jacobian matrix** $df(x)$. In the case of the curve $r : \mathbb{R} \to \mathbb{R}^n$, it was the **velocity** $dr(t) = r'(t)$. In the case of coordinate changes, the Jacobian matrix $df(x)$ was used to get the **volume distortion factor** $\det(df(x)) = \sqrt{\det(df^T df)}$. Today, we look at the case $m < n$. In particular at $m = 2, n = 3$. As in the case of curves, we use the letter r to describe the map. The image of a map $r : R \subset \mathbb{R}^m \to \mathbb{R}^n$ is then a **m-dimensional surface** in $\mathbb{R}^n$. The **distortion factor** $||dr||$ defined as $||dr||^2 = \det(dr^T dr)$ will be used later to compute **surface area**. [1]

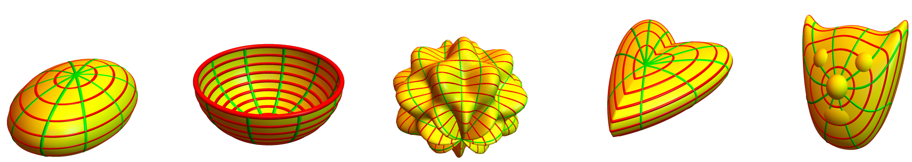

Figure 1. An ellipsoid, half an ellipsoid, a bulb, a heart and a cat.

11.2. We mostly discuss here the case $m = 2$ and $n = 3$, as we ourselves are made of two-dimensional surfaces, like cells, membranes, skin or tissue. A map $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$, written as $r(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ defines a two-dimensional surface. In order to save space, we also just write $r(u,v) = [x(u,v), y(u,v), z(u,v)]$. In computer graphics, the r is called uv**-map**. The uv-plane is where you draw a texture. The map r places it onto the surface. In geography, the map r is called (surprise!) a **map**. Several maps define an **atlas**. The curves $u \to r(u,v)$ and $v \to r(u,v)$ are called **grid curves**.

[1] Distinguish $||A||^2 = \det(A^T A)$ and $|A|^2 = \operatorname{tr}(A^T A)$ in $M(n,m)$. They only agree for $m = 1$.

11.3. The parametrization $r(\phi,\theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ produces the **sphere** $x^2+y^2+z^2 = 1$. The full sphere has $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$. By modifying the coordinates, we get an **ellipsoid** $r(\phi,\theta) = [a\sin(\phi)\cos(\theta), b\sin(\phi)\sin(\theta), c\cos(\phi)]$ satisfying $x^2/a^2+y^2/b^2+z^2/c^2 = 1$. By allowing a, b, c to be functions of ϕ, θ we get "bumpy spheres" like $r(\phi,\theta) = (3+\cos(3\phi)\sin(4\theta))[\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$.

11.4. Planes are described by linear maps $r(x) = Ax + b$ with $A \in M(3,2)$ and $b \in M(3,1)$. The Jacobian map is $dr = A$. Let r_u, r_v be the two column vectors of A. Actually, r_u is a short cut for $\partial_u r(u,v)$, which is the velocity vector of the **grid curve** $u \to r(u,v)$.

11.5. An example is the parametrization $r(u,v) = [u+v-1, u-v+3, 3u-5v+7]$ In this case $b = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$, $r_u = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ $r_v = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$ and $A = dr = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & -5 \end{bmatrix}$. We see $A^TA = \begin{bmatrix} 11 & -15 \\ -15 & 27 \end{bmatrix}$ which has determinant 72. We also have

$$|r_u \times r_v|^2 = |\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}|^2 = |\begin{bmatrix} -2 \\ 8 \\ -2 \end{bmatrix}|^2 = 72$$

11.6. The previous computation suggests a relation between the normal vector and the fundamental form $g = dr^T dr$. In three dimensions, the distortion factor of a parametrization $r : \mathbb{R}^2 \to \mathbb{R}^3$ can indeed always be rewritten using the cross product:

Theorem: $\det(dr^T dr) = |r_u \times r_v|^2$.

Proof. As $dr^T dr = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$, the identity is the **Cauchy-Binet identity** $|r_u \times r_v|^2 = |r_u|^2|r_v|^2 - |r_u \cdot r_v|^2$ which boils down to $\sin^2(\theta) = 1 - \cos^2(\theta)$, where θ is the angle between r_u and r_v. This is the angle between the grid curves you see on the pictures.

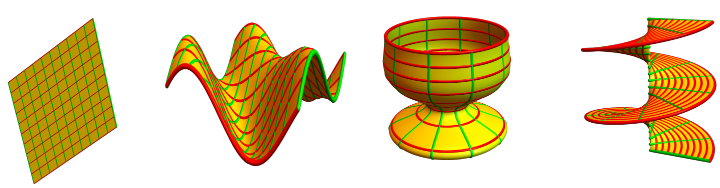

Figure 2. A plane, graph, surface of revolution and helicoid.

Examples

11.7. For the **unit sphere** $r(\phi,\theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ and $A = dr$:

$$g = A^TA = \begin{bmatrix} \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi) \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \\ \cos(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) \\ -\sin(\phi) & 0 \end{bmatrix}$$

This is $g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\phi) \end{bmatrix}$ and $\sqrt{\det(g)} = \sin(\phi)$ is the distortion factor.

11.8. An important class of surfaces are **graphs** $z = f(x,y)$. Its most natural parametrization is $r(x,y) = [x, y, f(x,y)]$, where the map r just lifts up the bottom part to the elevated version. An example is the elliptic paraboloid $r(x,y) = [x, y, x^2 + y^2]$ and the hyperbolic paraboloid $r(x,y) = [x, y, x^2 - y^2]$. We could of course have written also $r(u,v) = [u, v, u^2 - v^2]$.

11.9. A **surface of revolution** is parametrized like $r(\theta, z) = [g(z)\cos(\theta), g(z)\sin(\theta), z]$. Note that we can use any variables. In this case, $u = \theta, v = z$ are used. An example is the **cone** $r(\theta, z) = [z\cos(\theta), z\sin(\theta), z]$ or the **one-sheeted hyperboloid** $r(\theta, z) = [\sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta), z]$.

11.10. The **torus** is in cylindrical coordinates given as $(r-3)^2 + z^2 = 1$. We can parametrize this using the polar angle θ and the polar angle centered at center of the circle as $r(\theta, \phi) = [(3+\cos(\phi))\cos(\theta), (3+\cos(\phi))\sin(\theta), \sin(\phi)]$. Both angles θ and ϕ go from 0 to 2π. We see now also the relation with the **toral coordinates**.

11.11. The **helicoid** is the surface you see as a staircase or screw. The parametrization is $r(\theta, p) = [p\cos(\theta), p\sin(\theta), \theta]$. How can we understand this? The key is to look at grid curves. If $p = 1$, we get a curve $r(\theta) = [\cos(\theta), \sin(\theta), \theta]$ which we had identified as a **helix**. On the other hand, if you fix θ, then you get lines.

11.12. Side remark. The **first fundamental form** $g = dr^T dr$ is also called a **metric tensor**. In **Riemannian geometry** one looks at a manifold M equipped with a metric g. The simplest case is when g comes from a parametrization, as we did here. In physics, we know that it is **mass** which deforms space-time. The quantity $||g||^2 = \det(g)$ is a multiplicative analogue of $|g|^2 = \mathrm{tr}(g)$. For an invertible positive definite square matrix A, we will later see the identity $\log\det(A) = \mathrm{tr}\log(A)$ which illustrates how both determinant and trace are pivotal numerical quantities derived from a matrix. Trace is **additive** because of $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and determinant is **multiplicative** $\det(AB) = \det(A)\det(B)$ as we will see later.

11.13. To summarize, we have seen so far that there are two fundamentally different ways to describe a manifold. The first is to write it as a level surface $f = c$ which is a **kernel** of a map $g(x) = f - c$. A second is to write it as the **image** of some map r.

Illustration

Figure 3. "Veritas on Earth and the Moon" theme (rendered in Povray).

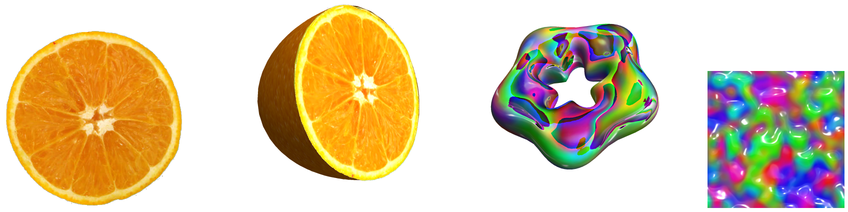

FIGURE 4. A fruit and math-candy© math-candy.com (rendered in Mathematica)

HOMEWORK

Problem 11.1: Parametrize the upper part of the two sheeted hyperboloid $x^2 + y^2 - z^2 = -1, z > 0$ in two different ways:
a) as a surface of revolution b) as a graph $z = f(x, y)$.

Problem 11.2: a) Parametrize the plane $x + 2y + 3z - 6 = 0$ using a map $r : \mathbb{R}^2 \to \mathbb{R}^3$. b) Now find the matrix $A = dr$ and compute $g = A^T A$ as well as the distortion factor $\sqrt{\det(A^T A)}$. c) Also compute r_u, r_v and $r_u \times r_v$ and then compute $|r_u \times r_v|$. You should get the same number.

Problem 11.3: Given a parametrization $r(\theta, \phi) = [(7 + 2\cos(\phi))\cos(\theta), (7 + 2\cos(\phi))\sin(\theta), 2\sin(\phi)]$ of the 2-torus, find the implicit equation $g(x, y, z) = 0$ which describes this torus.

Problem 11.4: Parametrize the hyperbolic paraboloid $z = x^2 - y^2$. What is the first fundamental form $g = dr^T dr$ which is $g = \begin{bmatrix} r_x \cdot r_x & r_x \cdot r_y \\ r_y \cdot r_x & r_y \cdot r_y \end{bmatrix}$?. What is the distortion factor $\sqrt{\det(g)}$?

Problem 11.5: The matrix $g = dr^T dr$ is also called the **first fundamental form**. If $r : \mathbb{R}^4$ to $\mathbb{R}^4$ is a parametrization of **space time** then g is the **space time metric tensor**. The matrix entries of g appear in **general relativity**. Now for some reasons, physics folks use Greek symbols to access matrix entries. They write $g_{\mu\nu}$ for the entry at row μ and column ν. This appears for example in the **Einstein field equations**

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} .$$

Find the general solution of this equation. Just kidding. We just want you to look up the equations and tell from each of the variables, what it is called and whether it is a matrix, a scalar function or a constant.