Unit 15: Contradiction and Deformation

Seminar

15.1. We have already seen one proof technique, the **"method of induction."** Other proofs were done either by **direct computations** or by **combining already known theorems or inequalities**. Today, we look at two new and fundamentally different proof techniques. The first is the method **"by contradiction."** The second method is the **"method of deformation."** Both methods are illustrated by a theorem.

15.2. The first theorem is one of the earliest results in mathematics. It is the **Hypassus theorem** from 500 BC. It was a result which shocked the Pythagoreans so much that Hypassus got killed for its discovery. That is at least what the rumors tell.

Theorem: The diagonal of a unit square has irrational length.

Proof. Assume the statement is false and the diagonal has rational length p/q. Then by Pythagoras theorem $2 = p^2/q^2$ or $2q^2 = p^2$. By the fundamental theorem of arithmetic, the left hand side has an odd number of factors 2, the right hand side an even number. This is a contradiction. The assumption must have been wrong.

15.3.

Problem A: Prove that the cube root of 2 is irrational.

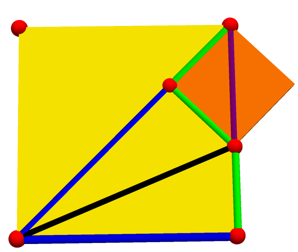

FIGURE 1. $\sqrt{2}$ is irrational. Start by assuming the side length and diagonal of the large yellow square are integers. Conclude that for the strictly smaller orange square, the side length and diagonal are integers.

15.4. Note that the proof relied on the fundamental theorem of arithmetic which assured that every integer has a unique prime factorization.

Problem B: Figure (1) is a geometric proof by contradiction which does not need the fundamental theorem of arithmetic. Complete the proof.

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15.5. Proofs by contradiction can be dangerous. A flawed proof can "assume the contrary, mess around with arguments, make a mistake somewhere and get a contradiction. QED". Better than a proof by contradiction is a constructive proof.

15.6. Here is a non-constructive proof which is amazing:

Theorem: There exist two irrational x, y such that x^y is rational.

Proof: there are two possibilities. Either $z = \sqrt{2}^{\sqrt{2}}$ is irrational or not. In the first case, we have found an example where $x = y = \sqrt{2}$. In the second case, take $x = z$ and take $y = \sqrt{2}$. Now $x^y = \sqrt{2}^2 = 2$ is rational and we have an example.

15.7. The second proof technique we see today is a **deformation argument**. To illustrate it, take a closed C^2 curve in $\mathbb{R}^2$ without self intersections. We have defined its curvature $\kappa(t)$ already. For curves in $\mathbb{R}^2$, define the **signed curvature** $K(t)$. If the curve parametrized so that $|r'(t)| = 1$ and $T(t) = [\cos(\alpha(t)), \sin(\alpha(t))]$, then $K(t) = \alpha'(t)$. Note that $\kappa(t) = |T'(t)| = |[-\sin(\alpha(t)), \cos(\alpha(t))]\alpha'(t)| = |K(t)|$. Now if we have a curve $r : [a, b] \to \mathbb{R}^2$, we can define the **total curvature** as $\int_a^b K(t)\, dt$. By the **fundamental theorem of calculus**, this total curvature is the change of the angle $\alpha(b) - \alpha(a)$. Now, if the curve is closed, the initial and final angles have to differ by a multiple of 2π. The **Hopf Umlaufsatz** tells that

Theorem: The total curvature of a simple closed curve is 2π or -2π.

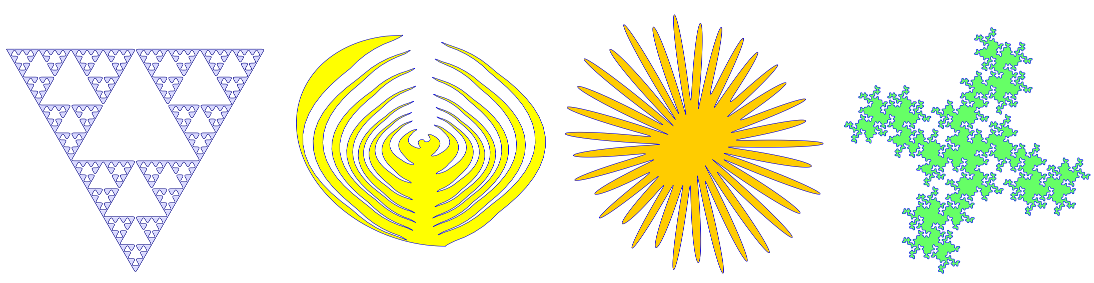

FIGURE 2. Four simple closed curves for which it is not obvious that the total curvature is 2π.

15.8.

Problem C: a) Why is the total curvature not always 2π?
b) Formulate out what happens in in Figure (3).

[1] for more explanation, see https://www.youtube.com/watch?v=Ih16BIoR9eM

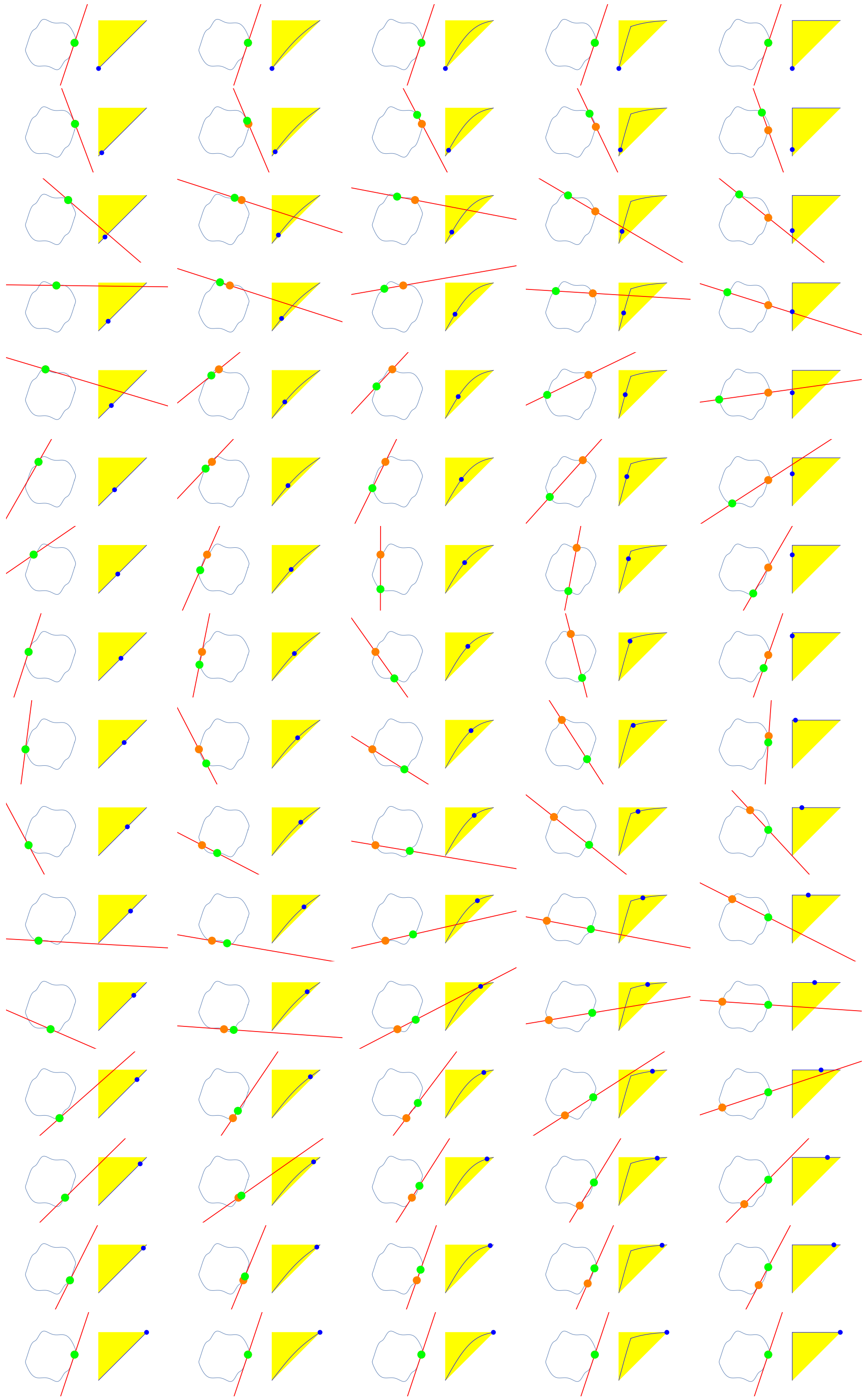

FIGURE 3. Hopf's deformation proof: each picture shows the line through $r(s), r(t)$ and to the right the parameter (s,t). In the left column, where $s = t$, we deal with the tangent turning. We have to show it turns by 2π. The next columns deform the situation where the path through the parameter square is changed. In the very right column, we twice turn the segment by π, in total 2π.

Homework

Exercises A-C are done in the seminar. This homework is due on Tuesday

Problem 15.1 Prove by contradiction that $\sqrt{12}$ is irrational.

Problem 15.2 Prove by contradiction that $\log_{10}(2)$ is irrational. $\log_{10}$ is the logarithm with respect to the base 10.

Problem 15.3 Verify the Hopf Umlaufsatz for a circle of radius 5, where $r(t) = \begin{bmatrix} 5\cos(t) \\ 5\sin(t) \end{bmatrix}$.

Problem 15.4 The Umlaufsatz generalizes to polygons. We can not assign a tangent to the vertices of the polygon but we can look at the deformation proof in the case when the parameters r, s are not equal. Look at the Hopf Umlaufsatz say in the case of a triangle with angles α, β, γ. What does the deformation argument tell then?

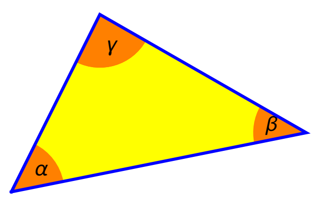

Figure 4. Can you adapt the Hopf Umlaufsatz for triangles?

Problem 15.5 There is a variant of proof by contradiction which is **proof by infinite descent**. It was used in proving a special case of **Fermat's Last theorem**. This special result tells that the equation $r^2 + s^4 = t^4$ has no solution with positive r, s, t. Look up and write down the proof of this theorem.

Figure 5. Perre de Fermat: cropped from Foto by Didier Descouens: showing the Monument to Pierre de Fermat by Alexandre Falguière in Beaumont-de-Lomagne, Tarn-et-Garonne France.

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