Unit 18: Number Magic

SEMINAR

18.1. In this seminar, we see how calculus can help to compute things effectively and also hope to get insight into topics which are of more number theoretical nature. To find the cube root of 10 for example, we have

$$10^{1/3} \sim 8^{1/3} + \frac{2}{3 \cdot 8^{2/3}} = 2 + \frac{2}{12} = 2.1666\ldots.$$

The actual value is 2.15443.

18.2.

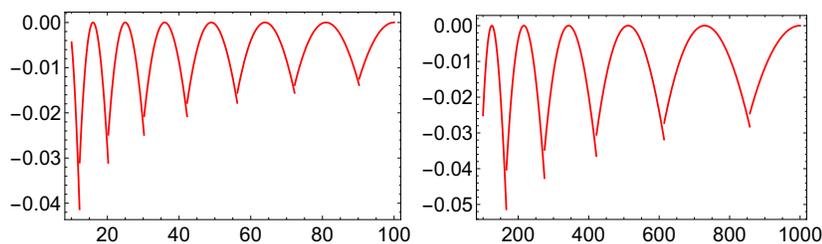> **Problem A:** Find the cube root of 999999 using linear approximation.

FIGURE 1.   The error of the linear approximation when computing square roots and cube roots is in the 5 percent range.

18.3. In the exam problem 1, you have proven $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$. This is a special case of the **geometric series formula**

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a} .$$

Of course, we could also prove this formula by induction. Better do it directly:

> **Problem B:** Verify the geometric series formula by multiplying with $1 - a$.

18.4. These were all finite sums but seeing the pattern allows us to **take a limit** and compute the infinite series:

> **Problem C:** For which a is $1 + a + a^2 + a^3 + ... = \frac{1}{1-a}$ valid?

18.5. Recall the definition of Taylor series and answer the following trick question:

> **Problem D:** What is the Taylor series of $f(x) = \frac{1}{(1-x)}$ at $x_0 = 0$?

18.6. How can you get from the last exercise the following identity?

> **Problem E:** $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$.

18.7. We can also differentiate to verify the formula $\sum_{n=1}^{\infty} nx^{n-1} = 1/(1-x)^2$ and so

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \ .$$

This function is called **Li$_{-1}(x)$**.

> **Problem F:** What is the numerical value of
> $$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots \ .$$

18.8. How come that great number theorists like Leonard Euler or Godfrey Hardy were also masters in calculus? The reason is that many results of number theoretic nature have intimate relations with calculus. Lets look at the following problem:

> **Problem G:** What is the value of the Leibniz series
> $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \ .$$

18.9. Hint: compute first the Taylor series of $f(x) = \arctan(x)$ using the Taylor series of $1/(1+x^2)$ (the later is a geometric series), then evaluate f at $x = 1$.

18.10.

$$\text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \cdots$$

is called the **poly logarithm function**. For $s = 0$ it is Problem D, for $s = 1$ it is problem E, for $s = -1, x = 1/2$ it is problem F. While in calculus, we might be more interested in the function as a function of x, **number theorists** are more interested in the function as a function of s and s is complex. In the case $x = 1$, one gets the **Riemann zeta function** $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$.

Problem H: What does the Riemann hypothesis say?

18.11. The **Euler golden key** relates ζ with primes:

Theorem: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1}$.

18.12.

Problem I: Verify the Euler golden key identity.

18.13. First verify (maybe look at Problem C) that for a single prime p

$$\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots$$

which is the sum over all $\frac{1}{n^s}$, where n has only prime factors p. Then look at the product of these for two primes p, q and see that this is the sum over all $\frac{1}{n^s}$ where n has only prime factors p and q.

18.14. The **Goldbach conjecture** tells that every even number larger than 2 is the sum of two primes. What is the relation with calculus? Define $g(x) = (f(x))^2$ with

$$f(x) = \sum_p \frac{x^p}{p!} = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.$$

Problem J: Goldbach is equivalent to $g^{(n)}(x) > 0$ for all even $n > 2$.

HOMEWORK

Exercises A-F are done in the seminar. This homework is due on Tuesday

Problem 18.1 The function f defined by $f(x) = e^{-1/x}$ for $x > 0$ and 0 for $x \leq 0$ is smooth and that all derivatives at 0 are zero. Check $f'(0), f''(0), f'''(0) = 0$.
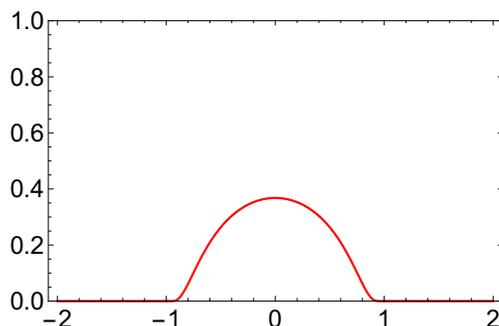

FIGURE 2. The function $f(x) = e^{-1/x}$ allows to define a **smooth bump function** $b(x)$ which is zero outside a ball of radius r.

Problem 18.2 Why is $b(x) = f(r^2 - |x|^2)$ a bump function?

Problem 18.3 The series
$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$
a long history. Research it a bit. Especially: What is the value? Who found this problem first? What is the name of the problem?

Problem 18.4 By looking it up, give an explanation why
$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots$$
can be assigned a finite value. You can also look up its value $-1/12$ with Mathematica $Zeta[-1]$. How is such a finite value possible?

Problem 18.5 You can practice computing square roots of numbers between 1 and 100 by linear approximation in your head. For example, if somebody asks you to compute $\sqrt{20}$ you would immediately tell $4 + 4/(2 \cdot 4) = 4.5$. The actual result is 4.472... You you could also get $5 - 5/(2 \cdot 5) = 4.5$. Find another non-square integer in 1 to 100 for which these two estimates agree. (There are a couple of them).