Unit 19: Extrema

Lecture

19.1. All functions are assumed here to be in C^2. It all starts with an observation going back to Pierre de Fermat:

> **Theorem:** If x_0 is a maximum of $f : \mathbb{R}^m \to \mathbb{R}$, then $\nabla f(x_0) = 0$.

Proof. We prove by contradiction. Assume $\nabla f(x_0) \neq 0$, define the vector $v = \nabla f(x_0)$ and look at $g(t) = f(x_0 + tv)$, which is a function of one variable. By the chain rule, it satisfies $g'(0) = \nabla f(x_0 + 0v) \cdot v = |\nabla f|^2 > 0$. This means that $f(x_0 + tv) > 0$ for small $t > 0$. The point x_0 can not have been maximal. This is a **contradiction**. QED.

19.2. A point x with $\nabla f(x) = 0$ is called a **critical point** of f. By the Taylor formula, we have at a critical point x_0 the quadratic approximation $Q(x) = f(x_0) + (x - x_0)^T H(x_0)(x - x_0)/2$, where $H(x_0)$ is the **Hessian matrix**

$$H(x_0) = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_m} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_m} \\ \cdots & \cdots & \cdots & \cdots \\ f_{x_mx_1} & f_{x_mx_2} & \cdots & f_{x_mx_m} \end{bmatrix} .$$

19.3. As in one dimension, having a critical point does not assure that a point is a local maximum or minimum. The second derivative test in single variable calculus assures that if $f'(x_0) = 0, f''(x_0) > 0$, we have a local minimum and if $f'(x_0) = 0, f''(x_0) < 0$, we have a local maximum. If $f''(x_0) = 0$, we can not say anything without looking at higher derivatives.

19.4. A matrix A is called **positive definite** if $v \cdot Av > 0$ for all vectors $v \neq 0$. It is called **negative definite** if $v \cdot Av < 0$ for all vectors $v \neq 0$. A diagonal matrix with positive diagonal entries is positive definite. In the following statements, we assume x_0 is a critical point.

19.5. We say x_0 is a **local maximum** of f if there exists $r > 0$ such that $f(x) \leq f(x_0)$ for all $|x - x_0| < r$. We say, it is a **local minimum** of f if $f(x) \geq f(x_0)$ for all $|x - x_0| < r$. How can we check whether a point is a local maximum or minimum?

> **Theorem:** Assume $\nabla f(x_0) = 0$. If $H(x_0)$ is positive definite, then x_0 is a local minimum. If $H(x_0)$ is negative definite, then x_0 is a local maximum.

19.6. Proof: as $\nabla f(x_0) = 0$, the quadratic approximation at x_0 is $Q(x) = f(x_0) + H(x_0)v \cdot v/2 > f(x_0)$ for small non-zero $v = x - x_0$ and Hessian H. The analogue statement for the minimum can be deduced by replacing f with $-f$.

19.7. Let us look at the case, where $f(x, y)$ is a function of two variables such that $f_x(x_0, y_0) = 0$ and $g_x(x_0, y_0) = 0$. The Hessian matrix is

$$H(x_0, y_0) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} .$$

In this two dimensional case, we can classify the critical points if the determinant $D = \det(H) = f_{xx}f_{yy} - f_{xy}^2$ of H is non-zero. The number D is also called the **discriminant** at a critical point.

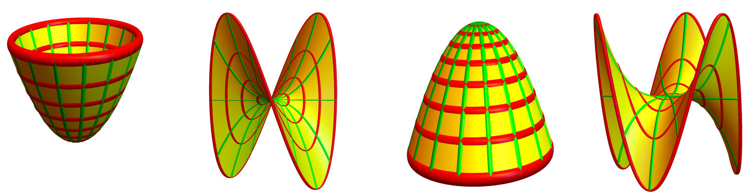

FIGURE 1. $f = x^2 + y^2$ gives a minimum, $f = -x^2 - y^2$ a maximum and $f = x^2 - y^2$ a saddle. The case $f = x^2y - yx^2$ is not Morse.

19.8. We say (x_0, y_0) is a **Morse point**, if (x_0, y_0) is a critical point and the determinant is non-zero. A C^2 function is a **Morse function** if every critical point is Morse. Examples of Morse functions are $f(x, y) = x^2 + y^2$, $f(x, y) = -x^2 - y^2$ and $f(x, y) = x^2 - y^2$. The last case is called a **hyperbolic saddle**. In general, a critical point is a hyperbolic saddle if $D \neq 0$ and if it is neither a maximum nor a minimum. Here is the **second derivative test** in dimension 2:

Theorem: Assume $f \in C^2$ has a critical point (x_0, y_0) with $D \neq 0$.
If $D > 0$ and $f_{xx} > 0$ then (x_0, y_0) is a local minimum.
If $D > 0$ and $f_{xx} < 0$ then (x_0, y_0) is a local maximum.
If $D < 0$ then (x_0, y_0) is a hyperbolic saddle.

19.9. Proof. After translation $(x, y) \to (x - x_0, y - y_0)$ and replacing f with $f - f(x_0, y_0)$, we have $(x_0, y_0) = (0, 0)$ and $f(0, 0) = 0$. At the critical point, the quadratic approximation is now

$$Q(x, y) = ax^2 + 2bxy + cy^2 .$$

This can be rewritten as $a(x+\frac{b}{a}y)^2+(c-\frac{b^2}{a})y^2 = a(A^2+DB^2)$ with $A = (x+\frac{b}{a}y), B = b^2/a^2$ and discriminant D. If $a = f_{xx} > 0$ and $D > 0$ then $c - b^2/a > 0$ and the function has positive values for all $(x, y) \neq (0, 0)$. The point $(0, 0)$ is then a minimum. If $a = f_{xx} < 0$ and $D > 0$, then $c - b^2/a < 0$ and the function has negative values for all $(x, y) \neq (0, 0)$ and the point (x, y) is a local maximum. If $D < 0$, then f takes both negative and positive values near $(0, 0)$. QED

19.10. One can ask, why f_{xx} and not f_{yy} is chosen. It does not matter, because if $D > 0$, then both f_{xx} and f_{yy} need to be non-zero and have the same sign. Instead of f_{xx}, one could also have pick the more natural **trace** $\mathrm{tr}(H)$. It is invariant under coordinate changes similarly as the determinant D. The discriminant D happens also to be the **Gauss curvature** of the surface at the point.

19.11. In higher dimensions, the situation is described by the **Morse lemma**. It tells that near a critical point there is a coordinate change ϕ such that $g(x) = f(\phi(x))$ is a quadratic function $f(x) = B(x - x_0) \cdot (x - x_0)$ where B is diagonal with entries $+1$ or -1. Critical point can then be given a **Morse index**, the number of entries -1 in B. The Morse lemma is actually a theorem (theorems are more important than lemmata=helper theorems)

> **Theorem:** Near a Morse critical point x_0 of a C^2 function f, there is a coordinate change $\phi : \mathbb{R}^m \to \mathbb{R}^m$ such that $g(x) = f(\phi(x)) - f(x_0)$ is
> $$g(x) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2 .$$

19.12. Proof. We use induction with respect to m. **(i) Induction foundation.** For $m = 1$, the result tells that for a Morse critical point, the function looks like $y = x^2$ or $y = -x^2$. First show that if $f(0) = f'(0) = 0, f''(0) \neq 0$, then $f(x) = x^2 h(x)$ or $f(x) = -x^2 h(x)$ for some positive C^2 function h. Proof. By a linear coordinate change we assume $x_0 = 0$ and $f(0) = 0$. There exists then $g(x)$ such that $f(x) = xg(x)$: it is $g(x) = f(x)/x$ for $x \neq 0$ and is in the limit $x \to 0$ the value of $\lim_{x\to 0}(f(x) - f(0))/x = f'(0)$. By the product rule, $f'(x) = g(x) + xg'(x)$ with $g(0) = 0$. Because $f'(0) = g(0) = 0$ can define $f(x)/x^2$ for $x \neq 0$ and take the limit $x \to 0$, because by applying Hôpital twice, the limit is $f''(0)$. The coordinate change is now given by a function $y = \phi(x)$ satisfying $g(x, y) = y\sqrt{h(y)} = x$. Implicit differentiation gives $g_y(0, 0) = \sqrt{h(y)} \neq 0$ so that by the implicit function theorem $y(x)$ exists.
(ii) Induction step $m \to m+1$**:** we first note that Taylor for C^2 with remainder term implies that $f(x_1, \dots, x_n) = \sum_{i,j} x_i x_j h_{ij}(x_1, \dots, x_n)$ with some continuous functions h_{ij}. Furthermore, the function value $h_{ij}(0) = f_{x_i x_j}(0) = H_{ij}(0)$ are the coordinates of the Hessian. Apply first a rotation so that $h_{11} \neq 0$. Now look at x_1 and keep the other coordinates constant. As in (i), find a coordinate change ϕ such that $f(\phi(x)) = \pm x_1^2 + g(x_2, \dots, x_m)$, where g inherits the properties of f [1], but is of one dimension less. By induction assumption, there is a second coordinate change such that $g(\psi(x)) = x_2^2 - \cdots - x_l^2 + x_{l+1}^2 + \cdots + x_m^2$. Combining ϕ and ψ produces the Morse normal form.

Examples

19.13. Q: Classify the critical points of $f(x, y) = x^3 - 3x - y^3 - 3y$. A: As $\nabla f(x, y) = [3x^2 - 3, -3y^2 + 3]^T$, the critical points are $(1, 1)$,$(-1, 1)$,$(1, -1)$ and $(-1, -1)$. We compute $H(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$. For $(1, 1)$ and $(-1, -1)$ we have $D = -4$ and so saddle points. For $(-1, 1)$, we have $D = 4, f_{xx} = -2$, a local max. For $(1, -1)$ where $D = 4, f_{xx} = 2$ we have a local min.

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[1] This will be more clear after having seen more linear algebra

Homework

Problem 19.1: Classify the critical points of the **area 51** function

$$f(x, y) = x^{51} - 51x - y^{51} + 51y$$

using the second derivative test. This function is classified.

Problem 19.2: The function $f(x, y) = 2x^3 + 2y^3 - 3x^2y^2$ is called the "happy function". Find and classify its extrema.
This function is not Morse as for one of the critical points, the discriminant D is zero. We want you nevertheless to decide whether this point is a "local maximum" a "local minimum" or "neither of them".

Problem 19.3: Where on the parametrized surface $r(u, v) = [u^2, v^3, uv]$ is the temperature $T(x, y, z) = 12x + y - 12z$ minimal. Classify all the critical points of the function $f(u, v) = T(r(u, v))$. [If you have found the function $f(u, v)$, you can replace u, v again with x, y if you like to work with a function $f(x, y)$.]

Problem 19.4: Find all the critical points of the function $f(x, y, z) = (x - 1)^2 - y^2 + xz^2$. In each of the cases, find the Hessian matrix. We have not talked about eigenvalues yet, but they are numbers λ such that $Hv = \lambda v$ for some non-zero vector. One can find them by looking for the roots of the characteristic polynomial $\chi_H(\lambda) = \det(L - \lambda)$. You can calculate them on a computer. Find in each case the eigenvalues.

Problem 19.5: a) Find a function $f(x, y)$ with 3 maxima and 3 saddle points and one minimum.
b) You see below a contour map of a function of two variables. How many critical points are there? Is the function a Morse function?