LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 22: Double integrals

Lecture

22.1. Given a bounded region R in $\mathbb{R}^2$ and a continuous function $f(x, y) : R \to \mathbb{R}$, define the Riemann integral $I = \iint_R f(x, y)\, dA$ as the $n \to \infty$ limit of

$$I_n = \frac{1}{n^2} \sum_{(i/n, j/n) \in R} f(\frac{i}{n}, \frac{j}{n}) .$$

The bounded region R is a defined as closed subset of $\mathbb{R}^2$ bound by finitely many differentiable curves $R = \{g_1 \leq c_1, \dots g_k \leq c_k\}$. As already in one dimension, the definition is designed to be independent of an orientation chosen on R. We are integrating like summing up a spread sheet. Just add up all entries. To justify that the limit exists, we again can use the Heine-Cantor theorem which tells that f is continuous on R if and only if it is uniformly continuous. This means there are numbers $M_n \to 0$ such that if $|(x_1, y_1) - (x_2, y_2)| \leq 1/n$, then $|f(x_1, y_1) - f(x_2, y_2)| \leq M_n$.

Theorem: For continuous f on a bounded region R, $\iint_R f dxdy$ exists.

22.2. Proof. In each cube $Q_{ij} = \{i/n \leq x \leq (i+1)/n, j/n \leq y \leq (j+1)/n\} \cap R$ define $a_{ij} = \min_{(x,y)\in Q_{ij}} f(x, y)$ and $b_{ij} = \max_{(x,y)\in Q_{ij}} f(x, y)$. Because the boundary was assumed to be given by a collection of curves which have finite total arc length L, the number of cubes Q_{ij} which intersect the boundary C is bounded by $4Ln$ (a curve of length 1 can maximally touch 4 squares). Define also $F = \max_{(x,y)\in R} |f(x, y)|$. We have with $K_n = 4LF/n$:

$$A_n - K_n \leq I_n \leq B_n + K_n ,$$

where $A_n = \sum_{i,j} a_{ij}/n^2$ and $B_n = \sum_{i,j} b_{ij}/n^2$ and K_n takes care of cubes Q_{ij} which intersect the boundary of R and so only contribute partially. Let I be the limsup of I_n. We have $B_n - A_n \leq M_n n^2/n^2 = M_n \to 0$ and $K_n \to 0$ as well so that $||I_n - I| \leq M_n + K_n \to 0$.

22.3. We rarely evaluate integrals using Riemann sums. Fortunately it is possible to reduce a double integral to single integrals. One can do that for **basic regions** which consist of two type of regions **"bottom to top" regions** $R = \{(x, y),\ a \leq x \leq b, c(x) \leq y \leq d(x)\}$ or **"left to right" regions** $R = \{(x, y),\ a(y) \leq x \leq b(y), c \leq y \leq d\}$. By cutting a general region into smaller pieces like intersecting with sufficiently small cubes $Q_{i,j}$ defined above, we can write any region as a union of such basic regions:

for large enough n, any $Q_{ij} \cap R$ us a basic region. Now we can define the integral in the first case as $\int_a^b [\int_{c(x)}^{d(x)} f(x,y)\, dy] dx$ and in the second case as $\int_c^d [\int_{a(y)}^{b(y)} f(x,y)\, dx] dy$. Is this the same? This is answered with Fubini, which we have already used. Let R be a rectangle $R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$. Here is the **Fubini theorem**:

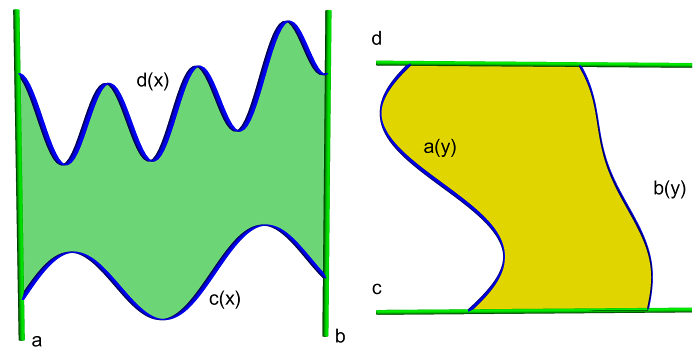

FIGURE 1. "Bottom to top" and "left to right" regions.

Theorem: $\int_R f(x,y)\, dA = \int_a^b [\int_c^d f(x,y)\, dy] dx = \int_c^d [\int_a^b f(x,y)\, dx] dy.$

22.4. Proof: first make a coordinate change to get $R = [0,1] \times [0,1]$, then cover R with n^2 cubes Q_{ij} of side length $1/n$. We have for every y a uniformly continuous function $x \to f(x,y)$ and for every x a uniformly continuous function $y \to f(x,y)$ and the constants M_n work for all: there is $M_n \to 0$ so that if $|x_1 - x_2| < 1/n$ and $|y_1 - y_2| < 1/n$, then $|f(x_1,y_1) - f(x_2,y_2)| \leq M_n$. Now use the notation $A \sim_c B$ if $|A - B| \leq c$ and get $\iint_R f(x,y) dA \sim_{M_n} \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{n} \sum_{j=0}^{n-1} f(i/n, j/n) \sim_{2M_n} \frac{1}{n} \sum_{i=0}^{n-1} \int_0^1 f(i/n, y)\, dy \sim_{3M_n}$ $\int_0^1 [\int_0^1 f(x,y)\, dy]\, dx$. Similarly, we can show $\iint_R f(x,y) dA \sim_{3M_n} \int_0^1 [\int_0^1 f(x,y)\, dx]\, dy$.

22.5. Without continuity, Fubini is false: the standard example is illustrated in Figure (2):

$$\frac{-\pi}{4} = \int_0^1 \int_0^1 \frac{(x^2 - y^2)}{(x^2 + y^2)^2}\, dydx \neq \int_0^1 \int_0^1 \frac{(x^2 - y^2)}{(x^2 + y^2)^2}\, dxdy = \frac{\pi}{4}\ .$$

Proof. $\int (x^2 - y^2)/(x^2 + y^2)^2\, dx = -x/(x^2 + y^2)$, $\int (x^2 - y^2)/(x^2 + y^2)^2\, dy = y/(x^2 + y^2)$. so that $\int_0^1 (x^2 - y^2)/(x^2 + y^2)^2\, dx = -1/(1 + y^2)$ and $\int_0^1 (x^2 - y^2)/(x^2 + y^2)^2\, dy = 1/(1 + x^2)$.

22.6. Integrals in higher dimensions are defined in the same way. We will cover the three dimensional case in particular later. Lets just add the definition for now. Given a m dimensional region R in $\mathbb{R}^m$ and a continuous $f : \mathbb{R}^m \to \mathbb{R}$, using the **multi-index notation** $x = (x_1, \dots, x_m), dx = dx_1 dx_2 \cdots dx_m$ and $i/n = (i_1/n, i_2/n, \dots, i_m/n)$ define

$$\int_R f(x) dx = \lim_{n \to \infty} \frac{1}{n^m} \sum_{\frac{i}{n} \in R} f(\frac{i}{n})\ .$$

A **region** is now a set $R = \{x \in \mathbb{R}^m \mid g_1(x) \leq c_1, \dots, g_k(x) \leq c_k\}$ where g_k are smooth functions. It is called **bounded** if there exists $\rho > 0$ such that $R \subset \{|x| \leq \rho\}$.

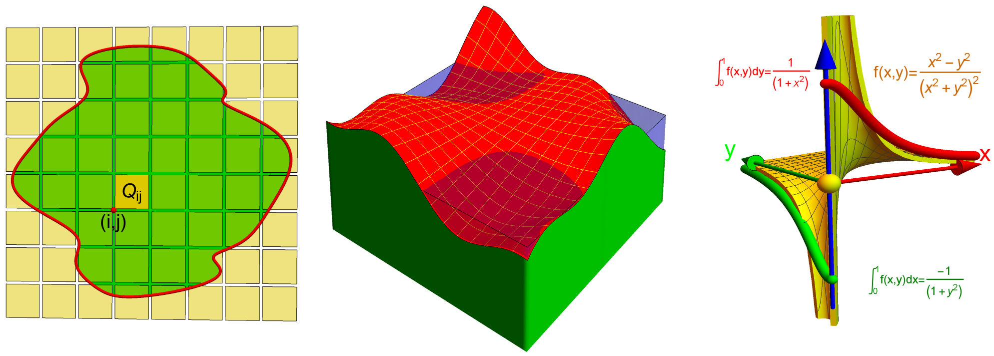

FIGURE 2. Integrating over a region via a Riemann integral. A double integral is a signed volume. Parts where $f < 0$ is negative volume. Fubini can fail, even if the two conditional integrals exist.

EXAMPLES

22.7. If $f(x, y) = 1$, then $\iint_R f(x, y)\ dxdy$ is the **area** of R. For example, if $\iint_{x^2+y^2\le 9}$ $8\ dxdy = 8\iint_{x^2+y^2\le 9} 1\ dxdy = 8\text{Area}(R) = 72\pi$.

22.8. We know from single variable calculus that $\int_a^b f(x)\ dx$ is the **signed area** under the curve of f. For $f(x) \ge 0$, where it is the area, we can write this as $\int_a^b \int_0^{f(x)} 1\ dydx$. Note that as we have defined the integrals, the equivalence would be wrong if $f(x)$ is negative somewhere. It is the double integral which is the correct notion of area. **Example:** The area of the region bounded by the curve $y = 1/(1 + x^2)$, the curve $y = 0$ and the curve $x = -1$ and $x = 1$ is $\int_{-1}^1 \int_0^{1/(1+x^2)}\ dydx = \arctan(x)|_{-1}^1 = \pi/2$.

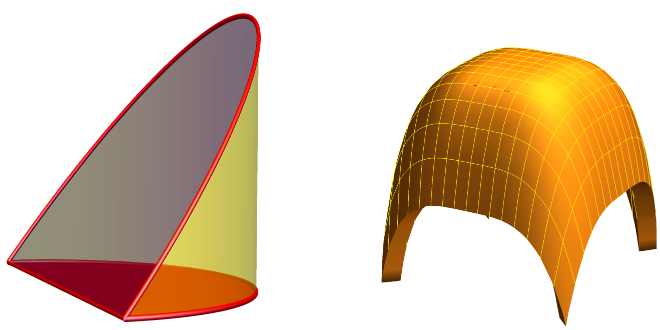

FIGURE 3.

22.9. The integral $\iint_R f(x, y)\ dxdy$ can be interpreted as the signed volume under the graph of f above the region R. Find the volume of the region bound by $z = 4 - 2x^4 - 2y^4$ and $z = 4 - 2x^2 - 2y^2$ and $-1 \le x \le 1$ and $-1 \le y \le 1$. Solution: $\int_0^1 \int_0^1 (4 - 2x^4 - 2y^4) - (4 - 2x^2 - 2y^2)\ dxdy = (4/15)^2$.

22.10. Problem. Find the area of a disc of radius a. Solution:

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 1\ dydx = \int_{-a}^{a} 2\sqrt{a^2 - x^2}\ dx\ .$$

Use **trig substitution** $x = a\sin(u), dx = a\cos(u)$, to get

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{a^2 - a^2\sin^2(u)}a\cos(u)\ du = \int_{-\pi/2}^{\pi/2} 2a^2\cos^2(u)\ du\ .$$

Using a double angle formula, this gives $a^2\int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u))}{2}\ du = a^2\pi$. We will next time compute this much more effectively.

22.11. Problem. Let R be the triangle $\{1 \geq x \geq 0, 0 \leq y \leq x\}$. Evaluate $\int\int_R e^{-x^2}\ dxdy$. **Solution.** We can not evaluate the integral directly because e^{-x^2} has no anti-derivative given in terms of elementary functions. But we can write the integral as $\int_0^1[\int_0^x e^{-x^2}\ dy]\ dx$

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2}\ .$$

Homework

Problem 22.1: Calculate the iterated integral $\int_0^1\int_x^{2-x}(x^2 - y)\ dydx$ in two ways, once as a "left to right" and once as a "bottom to top" integral.

Problem 22.2: Find the integral

$$\int_0^1\int_{\sqrt{y}}^{y^2}\frac{3x^7}{\sqrt{x}-x^2}\ dx\,dy\ .$$

Problem 22.3: Compute the area of the region bound by the ellipse $x^2/4^2 + y^2/9^2 = 1$ using trig substitution. (It is the "hardest problem in geometry", according to the comedy-drama "Rushmore", a movie from 1998).

Problem 22.4: Find the integral

$$\int_0^{\pi^2}\int_{\sqrt{y}}^{\pi}\frac{\sin(x)}{x^2}\ dxdy\ .$$

Problem 22.5: Find the volume of the hoof solid $x^2+y^2 \leq 1, 0 \leq z \leq x$. The hoof solid was considered by Archimedes already.