Unit 23: Substitution

LECTURE

23.1. If $\Phi : R \to S, \begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$ is a **coordinate change**, then the **distortion factor** was defined as $|d\Phi| = |\det(d\Phi)|$, where

$$d\Phi(u,v) = \begin{bmatrix} \partial_u x(u,v) & \partial_v x(u,v) \\ \partial_u y(u,v) & \partial_v y(u,v) \end{bmatrix}.$$

The **change of variable theorem** is the same in all dimensions. In the following proof, we assume that Φ is C^2. Because of Heine-Cantor, we know there exists $M_n \to 0$ with $|\frac{d^2}{dt^2}\Phi(u_0 + tv, v_0 + tw)| \leq M_n$ for $\sqrt{v^2 + w^2} \leq 1/n$ and all $(u_0, v_0) \in R$. [1]

Theorem: $\iint_R f(\Phi(u,v)) \, |d\Phi(u,v)| du dv = \iint_S f(x,y) \, dx dy.$

23.2. Proof. Cover S with cubes Q_{ij} as in the last lecture. Then

$$\iint_S f(x,y) \, dx dy = \sum_{Q_{ij}} \iint_{Q_{ij} \cap S} f(x,y) \, dx dy \sim \sum_{i,j} f(\frac{i}{n}, \frac{j}{n}) \frac{1}{n^2}.$$

The transformed squares $\Phi(Q_{ij})$ are close to the **parallelograms** $d\Phi(Q_{ij})$ which have area $|d\Phi(i/n, j/n)|/n^2$. Now make a quadratic Taylor expansion $\Phi(x,y) = \Phi(x_0, y_0) + d\Phi(x_0, y_0)(x - x_0, y - y_0) + d^2\Phi(x_0, y_0)(x - x_0, y - y_0)^2/2$ at $(x_0, y_0) = (i/n, j/n)$, where $|d^2\Phi(x_0, y_0)(x - x_0, y - y_0)^2| \leq M_n$. Let $F = \max_{(x,y) \in R}(|f(x,y)|)$. Applying in every direction, Taylor with remainder, we see

$$| \int_{\Phi(Q_{ij}) \cap S} f(x,y) \, dx dy - f(\Phi(\frac{i}{n}, \frac{j}{n}))|d\Phi(\frac{i}{n}, \frac{j}{n})|\frac{1}{n^2}| \leq \frac{M_n F}{n^2}.$$

As the number of squares hitting R is bound by $An^2 + 4Ln$ where A is the area of R and L is the length of the boundary of R, the sum of the non-linear errors is therefore bound by $(An^2 + 4Ln)M_n F/n^2$ which goes to zero for $n \to \infty$. QED.

23.3. Here is an example: If $\Phi : R = [0,1] \times [0, 2\pi] \to S = \{x^2 + y^2 \leq 1\}$ is given by $\Phi(r, \theta) = [r\cos(\theta), r\sin(\theta)]^T$. Then $d\Phi(r, \theta) = r$. If $f(x,y) = x^2 + y^2 = r^2$, then $\iint_R r^2 \, r dr \, d\theta = \iint_S x^2 + y^2 \, dx dy$. The first integral is $2\pi/4$.

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[1] For the C^1 case, see J. Schwartz, Mathematical Monthly 61, 1954, or P.D. Lax, Monthly 108, 2001

23.4. Let $\Phi : [0,1] \times [0,1] \to [0,1] \times [0,1]$ be given as $\Phi(x,y) = (y,x)$. Now $\det(d\Phi) = -1$ and $|d\Phi| = 1$. While we usually could ignore talking about orientation, it is evident here that the integrals considered so far, we do not care about the orientation of the space. If the change of coordinates switches the orientation, the resulting integral does not change.

23.5. The chain rule assures that combining two coordinate changes Φ, Ψ, gives a new coordinate change with $d(\Psi \circ \Phi)(x) = d\Psi(\Phi(x))d\Phi(x)$. For example if $\Psi(x,y) = [ax, by]^T$ and $\Phi(r,\theta) = [r\cos(\theta), r\sin(\theta)]^T$ changes into polar coordinates, then $\Psi(\Phi(r,\theta)) = [ar\cos(\theta), br\sin(\theta)]^T$. Now the image of $R = [0,1] \times [0,2\pi]$ is the ellipse $S = \{x^2/a^2 + y^2/b^2 \leq 1\}$ and the **area of the ellipse** is $A = \iint_R abr \, dr d\theta$ because $\det(d\Phi) = r$ and $\det(d\Psi) = ab$. The result is $\int_0^1 \int_0^{2\pi} abr \, d\theta dr = \pi ab$.
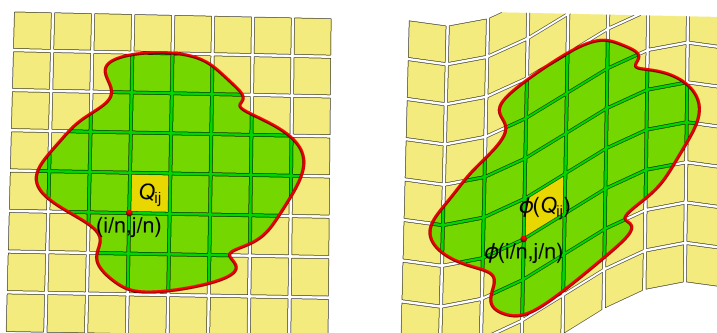

FIGURE 1. Coordinate change.

23.6. Preview: We will next week look at more general cases like $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$ of a parametrized surface, where the distortion factor is $|dr| = \sqrt{\det(dr^T dr)} = |r_u \times r_v|$ and the surface area is $\iint_R |r_u \times r_v| du dv = \iint_S 1 \, dA$.

23.7. The theorem generalizes **substitution** $\int_c^d f(\Phi(x))|\Phi'(x)| \, dx = \int_a^b f(x) \, dx$ if $\Phi(c) = a$ and $\Phi(d) = b$. We usually insist that Φ is monotonically increasing and write $u = \Phi(x), du = \Phi'(x)dx$ to get computations like in $\int_0^{\sqrt{\pi/2}} \sin(x^2)2xdx = \int_0^{\pi/2} \sin(u) \, du$, where $\Phi(x) = x^2$. As a hack, one can extend the formula to the case when Φ can decrease in which case the $[a,b]$ interval becomes the negative $[b,a]$ interval with $a < b$. Example: Let $\Phi(x) = 2 - 2x$ which has $\Phi' = -2$, then $\int_{1/2}^1 (2-2x)^2|(-2)|dx = \int_0^1 x^2 dx$. In single variable calculus, one can also work with the negative sign case and compute $\int_1^{1/2} (2-2x)^2(-2)dx$ which works if $\int_1^{1/2} = -\int_{1/2}^1$ but this is **not compatible** with the defined Riemann integral: we use **"spread-sheet" summation** and do not distinguish whether we add up the function values from left to right or from right to left. [2]

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[2]In single variable one can easily go from orientation independent 'Bosonic' integrals to 'Fermionic' integrals. In higher dimensions, one can then apply the derivative to anti-symmetric tensors. The switch "Bosonic" → "Fermionic" requires however to orient objects like curves or surfaces.

23.8. We can again look at the Fubini counter example $\iint_{x^2+y^2\leq 1}(x^2 - y^2)/(x^2 + y^2)^2 \; dxdy = \int_0^1 \int_0^{2\pi} \cos(2\theta)/r \; d\theta dr = 0$. We can not change the order of integration as we can not integrate $\int_0^1 1/r \; dr$. The trouble also continues in the new coordinate system and it is even more dramatic.

23.9. If $\Phi : x \to Ax$ and $\Psi : x \to Bx$ are two linear coordinate changes then $\Psi \circ \Phi = BA$ is the matrix product and the chain rule tells $|d(\Psi \circ \Phi)| = |\det(AB)|$ which agrees with the product $|d\Psi||d\Phi| = |\det(A)||\det(B)|$. We can do the verification of the **Cauchy-Binet** formula $\det(AB) = \det(A)\det(B)$ directly. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, then $AB = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$ and you can check the determinant formula.

23.10. Here is a famous open problem about coordinate changes. It is called the **Jacobian conjecture**. It deals with polynomial coordinate changes, where $x(u,v)$ and $y(u,v)$ are **polynomials** in u, v.

> **Conjecture:** If Φ is polynomial and $|d\Phi|$ is constant different from zero, then Φ has a polynomial inverse.

One knows that if the conjecture is false, then there exists a counter example with integer polynomials and Jacobian determinant 1. The conjecture is open since at least 1939. An example of a coordinate transformation with determinant 1 and integer polynomials are **Hénon maps** from lecture 16. If $\Phi([u,v]^T) = [x,y]^T = [u^2 - u^4 - v, u]^T$, then $\Phi^{-1}([x,y]^T) = [y, y^2 - y^4 - x]^T$.

<div align="center">EXAMPLES</div>

23.11. Problem: What is the area of the image $S = \Phi(R)$ if $\Phi([u,v]) = [u^2 - v^2 + 1, 2uv + 2]^T$ and $R = \{1 \leq u \leq 3, 0 \leq v \leq 1\}$. (This is $\Phi(z) = z^2 + c$ with $c = 1 + 2i$ in the complex). We have $d\Phi(u,v) = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$ and $|d\Phi(u,v)| = 4u^2 + 4v^2$. We see from the change of variables formula that the area is $\int_0^1 \int_1^3 4u^2 + 4v^2 \; dudv = 112/3$.

23.12. Problem: What is the **moment of inertia** $\iint_R x^2 + y^2 \; dxdy$, where R is the polar region given in polar coordinates as $r \leq 2 + \sin(3\theta)$. Solution: using the polar coordinate change of variables Φ with $|d\Phi| = r$, we get $\int_0^{2\pi} \int_0^{2+\sin(3\theta)} r^2 \; r \; drd\theta = \int_0^{2\pi} (2 + \sin(3\theta))^4/4 \; d\theta$. We explain in class how to get the answer $227\pi/4$ quickly.

23.13. Problem: Here is a famous problem. It is so popular, that it even made it to Hollywood: compute $\iint_{\mathbb{R}^2} e^{-x^2-y^2} \; dxdy$. Solution: this problem looks difficult at first as we can not integrate with respect to x or y. The function e^{-x^2} has no elementary anti-derivative. This **improper integral** is doable in polar coordinates as it is $\int_0^{2\pi} \int_0^\infty e^{-r^2} r \; dr \; d\theta = \pi$. It is the inner part $\int_0^\infty e^{-r^2} r \; dr$ which is an improper integral. One deals with this by approximation. For every finite L we have $\int_0^L e^{-r^2} r \; dr = -e^{-r^2}/2|_0^L = 1/2 - e^{-L^2}/2$ This converges nicely to $1/2$ for $L \to \infty$. It follows (and that is the punch line) that $\int_{-\infty}^\infty e^{-x^2} \; dx = \sqrt{\pi}$.

Homework

Problem 23.1: Given a disk $R = \{x^2 + y^2 \leq 1\}$, we can make this into a probability space and define the **expectation** of a function f as $\mathrm{E}[f] = \iint_R f \, dxdy/\pi$. The expectation of the random variables $f(x, y) = x^n$ are examples of **moments**. Find $\mathrm{E}[x], \mathrm{E}[x^2], \mathrm{E}[x^3]$ and $E[x^4]$.

Problem 23.2: What is the volume of the solid bound by $z = f(x, y) = x^2 + y^2$ and $z = g(x, y) = 8 - x^2 - y^2$? You can write this as a double integral $\iint_R g(x, y) - f(x, y) \, dxdy$ over a suitable region.

Problem 23.3: The fidget spinner is so "2017" now. What is hot now is the **math 22 spinner** with 23 bearings! What is the moment of inertia $\int \int_G x^2 + y^2 \, dxdy$ of the **math 22 fidget spinner region** G given in polar coordinates as $1/2 \leq r \leq 2 + \cos(22\theta)$. To keep our bearings, we do not count the bearings.

Problem 23.4: Biologist **Piet Gielis** once patented polar regions in order to use them to describe biological shapes like cells, leaves, starfish or butterflies. Don't worry about violating patent laws when finding the area of the following butterfly $r(t) \leq |8 - \sin(t) + 2\sin(3t) + 2\sin(5t) - \sin(7t) + 3\cos(2t) - 2\cos(4t)|$. (It can produce butterflies in your stomach but there are some tricks to do that fast. Relax with the Math 22 fidget spinner for example!)
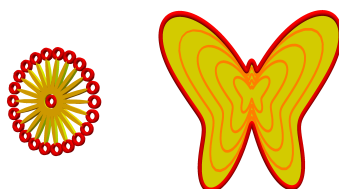

FIGURE 2. The math 22 spinner and the butterfly.

Problem 23.5: a) Prove the Jacobian conjecture for linear maps $\Phi(x) = Ax$, where A is a 2×2 matrix.
b) Find a linear coordinate change $\Phi(x, y)$ for which the Jacobian determinant is 1. It should be non-trivial in the sense, that we don't just want a diagonal matrix $d\Phi$.
c) Find a counter example of the Jacobian conjecture for cubic polynomials (just kidding). Find an example for the Jacobian conjecture where both polynomials are not linear!