LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 29: Line integrals

Lecture

29.1. A **vector field** F assigns to every point $x \in \mathbb{R}^n$ a vector $F(x) = [F_1(x), \dots, F_n(x)]^T$ such that every $F_k(x)$ is a continuous function. We think of F as a **force field**. Let $t \to r(t) \in \mathbb{R}^n$ be a curve parametrized on $[a, b]$. The integral

$$\int_C F \cdot \, dr = \int_a^b F(r(t)) \cdot r'(t) \, dt$$

is called the **line integral** of F along C. We think of $F(r(t)) \cdot r'(t)$ as **power** and $\int_C F \cdot \, dr$ as the **work**. Even so F and r are column vectors, we write in this lecture $[F_1(x), \dots, F_n(x)]$ and $r' = [x_1', \dots, x_n']$ to avoid clutter. Mathematically, $F : \mathbb{R}^n \to \mathbb{R}^n$ can also be seen as a coordinate change, we think about it differently however and draw a vector $F(x)$ at every point x.

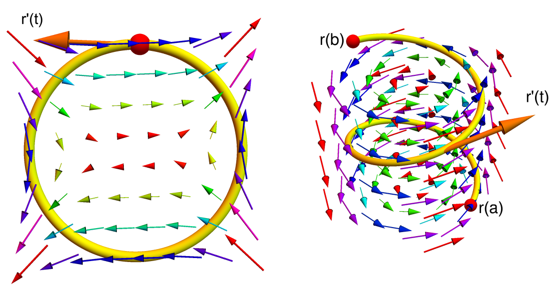

Figure 1. A line integral in the plane and a line integral in space.

29.2. If $F(x, y) = [y, x^3]$, and $r(t) = [\cos(t), \sin(t)]$ a circle with $0 \leq t \leq 2\pi$, then $F(r(t)) = [\sin(t), \cos^3(t)]$ and $r'(t) = [-\sin(t), \cos(t)]$ so that $F(r(t)) \cdot r'(t) = -\sin^2(t) + \cos^4(t)$. The work is $\int_C F \cdot \, dr = \int_0^{2\pi} -\sin^2(t) + \cos^4(t) \, dt = -\pi/4$. Figure 1 shows the situation. We go more against the field than with the field.

29.3. A vector field F is called a **gradient field** if $F(x) = \nabla f(x)$ for some differentiable function f. We think of f as the **potential**. The first major theorem in vector calculus is the **fundamental theorem of line integrals** for gradient fields in $\mathbb{R}^n$:

Theorem: $\int_a^b \nabla f(r(t)) \cdot r'(t) \, dt = f(r(b)) - f(r(a))$.

29.4. Proof: by the **chain rule**, $\nabla f(r(t)) \cdot r'(t) = \frac{d}{dt} f(r(t))$. The **fundamental theorem of calculus** now gives $\int_a^b \frac{d}{dt} f(r(t))\, dt = f(r(b)) - f(r(a))$. QED.

29.5. As a corollary we immediately get path independence

If C_1, C_2 are two curves from A to B then $\int_{C_1} F \cdot\, dr = \int_{C_2} F \cdot\, dr$,

as well as the closed loop property:

If C is a closed curve and $F = \nabla f$, then $\int_C F \cdot\, dr = 0$.

29.6. Is every vector field F a gradient field? Lets look at the case $n = 2$, where $F = [P, Q]$. Now, if this is equal to $[f_x, f_y] = [P, Q]$, then $P_y = f_{xy} = f_{yx} = Q_x$. We see that $Q_x - P_y = 0$. More generally, we have the following **Clairaut criterion**:

Theorem: If $F = \nabla f$, then $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j = 0$.

Proof: this is a consequence of the Clairaut theorem.

29.7. The field $F = [0, x]$ for example satisfies $Q_x - P_y = 1$. It can not be a gradient field. Now, if $Q_x - P_y = 0$ everywhere in the plane, how do we find the potential f?

Integrate $f_x = P$ with respect to x and add a constant $C(y)$.

Differentiate f with respect to y and compare f_y with Q. Solve for $C(y)$.

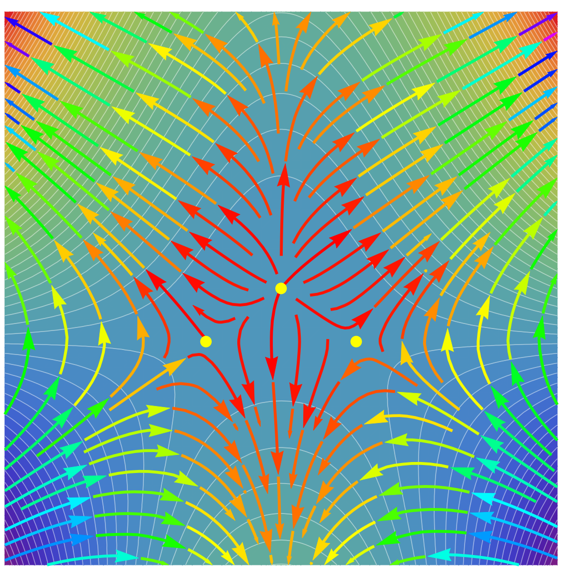

FIGURE 2. The vector field $F = \nabla f$ for $f(x, y) = y^2 + 4yx^2 + 4x^2$. We see the **flow lines**, curves with $r'(t) = F(r(t))$. Going with the flow **increases** f because $F(r(t)) \cdot r'(t) = |\nabla f(t)|^2$ is equal to $d/dt f(r(t))$.

29.8. Example: find the potential of $F(x, y) = [P, Q] = [2xy^2 + 3x^2, 2x^2y + 3y^2]$. We have $f(x, y) = \int_0^x 2xy^2 + 3x^2 \; dx + C(y) = x^3 + x^2y^2 + C(y)$. Now $f_y(x, y) = 2x^2y + C'(y) = 2x^2y + 3y^2$ so that $C'(y) = 3y^2$ or $C(y) = y^3$ and $f = x^3 + x^2y^2 + y^3$.

29.9. Here is a direct formula for the potential. Let C_{xy} be the straight line path which goes from $(0, 0)$ to (x, y).

Theorem: If F is a gradient field then $f(x, y) = \int_{C_{xy}} F \cdot \; dr$.

29.10. Proof: By the fundamental theorem of line integral, we can replace C_{xy} by a path $[t, 0]$ going from $(0, 0)$ to $(x, 0)$ and then with $[x, t]$ to (x, y). The line integral is $f(x, y) = \int_0^x [P, Q] \cdot [1, 0] dt + \int_0^y [P, Q] \cdot [0, 1] \; dt = \int_0^x P(t, 0) \; dt + \int_0^y Q(x, t) \; dt$. We see that $f_y = Q(x, y)$. If we use the path going $(0, 0)$ to $(0, y)$ and to (x, y) instead, the line integral is $f(x, y) = \int_0^y [P, Q] \cdot [0, 1] dt + \int_0^x [P, Q] \cdot [1, 0] \; dt = \int_0^y Q(0, t) \; dt + \int_0^x P(t, y) \; dt$. Now, $f_x = P(x, y)$. QED.

Examples

29.11. Find $\int_C [2xy^2 + 3x^2, 2x^2y + 3y^2] \cdot \; dr$ for a curve $r(t) = [t\cos(t), t\sin(t)]$ with $t \in [0, 2\pi]$. Answer: we found already $F = \nabla f$ with $f = x^3 + x^2y^2 + y^3$. The curve starts at $A = (1, 0)$ and ends at $B = (2\pi, 0)$. The solution is $f(B) - f(A) = 8\pi^3$.

29.12. If $F = E$ is an electric field, then the line integral $\int_a^b E(r(t)) \cdot r\,'(t) \; dt$ is an **electric potential**. In celestial mechanics, if F is the gravitational field, then $\int_a^b F(r(t)) \cdot r\,'(t) \; dt$ is a **gravitational potential** difference. If $f(x, y, z)$ is a temperature and $r(t)$ the path of a fly in the room, then $f(r(t))$ is the temperature, which the fly experiences at the point $r(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(r(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference.

29.13. A device which implements a non-gradient force field is called a **perpetual motion machine**. It realizes a force field for which the energy gain is positive along some closed loop. The **first law of thermodynamics** forbids the existence of such a machine. It is informative to contemplate the ideas which people have come up and to see why they don't work. We will look at examples in the seminar.

29.14. Let $F(x, y) = [P, Q] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]$. Its potential $f(x, y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1 + y^2/x^2) = P, f_y = (1/x)/(1 + y^2/x^2) = Q$. In the seminar you ponder the riddle that the line integral along the unit circle is not zero:

$$\int_0^{2\pi} [\frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)}] \cdot [-\sin(t), \cos(t)] \; dt = \int_0^{2\pi} 1 \; dt = 2\pi \; .$$

The vector field F is called the vortex.

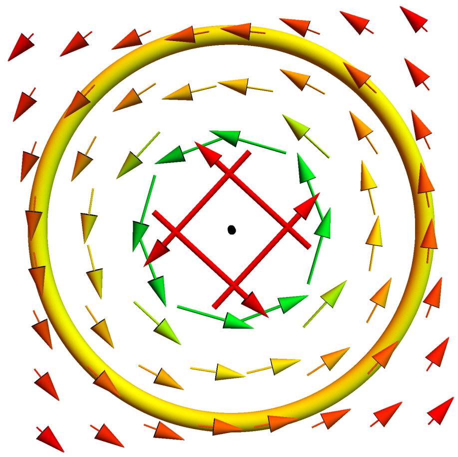

FIGURE 3. The vortex vector field has a singularity at $(0,0)$. All the curl is concentrated at $(0,0)$.

HOMEWORK

Problem 29.1: Let C be the space curve $r(t) = [\cos(t), \sin(t), \sin(t)]$ for $t \in [0, \pi/2]$ and let $F(x, y, z) = [y, x, 15]$. Calculate the line integral $\int_C F \cdot dr$.

Problem 29.2: What is the work done by moving in the force field $F(x, y) = [2x^3 + 1, 4\pi \sin(\pi y^4) y^3]$ along the quartic $y = x^4$ from $(-1, 1)$ to $(1, 1)$?

Problem 29.3: Let F be the vector field $F(x, y) = [-y, x]/2$. Compute the line integral of F along the curve $r(t) = [a\cos(t), b\sin(t)]$ with width $2a$ and height $2b$. The result should depend on a and b.

Problem 29.4: Archimedes swims around a curve $x^{22} + y^{22} = 1$ in a hot tub, in which the water has the velocity $F(x, y) = [3x^3 + 5y, 10y^4 + 5x]$. Calculate the line integral $\int_C F \cdot dr$ when moving from $(1, 0)$ to $(-1, 0)$ along the curve.

Problem 29.5: Find a closed curve $C : r(t)$ for which the vector field

$$F(x, y) = [P(x, y), Q(x, y)] = [xy, x^2]$$

satisfies $\int_C F(r(t)) \cdot r'(t)\, dt \neq 0$.

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