Unit 32: Stokes theorem

Lecture

32.1. Given a C^1 surface $S = r(G)$ in $\mathbb{R}^3$ and a differentiable vector field $F = [P, Q, R]$, we can form the **flux integral**

$$\iint_S F \cdot dS = \iint_G F(r(u,v)) \cdot r_u \times r_v \; dudv \; .$$

For $F = [P, Q, R]$, the **curl** is defined as $\nabla \times F = [R_y - Q_z, P_z - R_x, Q_x - P_y]$. The **Stokes theorem** tells that if $C = r(I)$ is the boundary of $S = r(G)$ and I is oriented so that G is to the left, then

Theorem: $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$.

32.2. Proof. The key is the **"important formula"**

$$\text{curl}(F)(r(u,v)) \cdot (r_u \times r_v) = F_u \cdot r_v - F_v \cdot r_u .$$

This is straightforward and done in class. Now define the field $\tilde{F}(u,v) = [\tilde{P}, \tilde{Q}] = [F(r(u,v)) \cdot r_u(u,v), F(r(u,v)) \cdot r_v(u,v)]$ in the uv-plane. The 2-dimensional curl of $\tilde{F}$ is $\tilde{Q}_u - \tilde{P}_v = F_u \cdot r_v - F_v \cdot r_u$ as we can see by using Clairaut $r_{uv} = r_{vu}$. The Stokes theorem is now a direct consequence of Green's theorem proven last time. QED. [1]

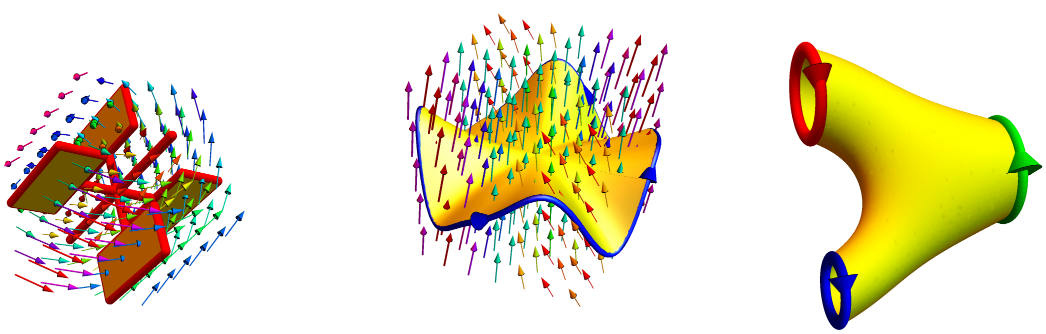

FIGURE 1. The paddle wheel measures curl. The boundary C has S "to the left". The pant surface illustrates a "cobordism".

[1]Mathematicians say: "we pulled back the field from $\mathbb{R}^3$ to $\mathbb{R}^2$ along the parametrization".

Examples

32.3. Problem: Compute the flux of $F(x,y,z) = [0,0,8z^2]^T$ through the upper half unit sphere S oriented outwards. **Solution:** we parametrize the surface as $r(u,v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)]^T$. Because $r_u \times r_v = -\sin(v)r$, this parametrization has the wrong orientation! We continue nevertheless and just change the sign at the end. We have $F(r(u,v)) = [0,0,8\cos^2(v)]^T$ so that

$$\int_0^{2\pi} \int_0^{\pi/2} -[0,0,8\cos^2(v)]^T \cdot [\cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)]^T \, dvdu \, .$$

The flux integral is $\int_0^{2\pi} \int_0^{\pi/2}$ - $8\cos^3(v)\sin(v)\, dvdu$ which is $2\pi \cdot 8\cos^4(v)/4|_0^{\pi/2} = -4\pi$. The flux with the outward orientation is $+4\pi$. We could **not** use the Stokes theorem here because we don't deal with the flux of the curl but the flux of F itself.

32.4. Problem: What is the value of $\int_C F \cdot dr$ if $F = [\sin(\sin(x)) + z^2, e^y + x^3 + y^2, \sin(y^2) + z^2]$ and C is the unit polygon $(0,0,0) \to (1,0,0) \to (1,1,0) \to (0,1,0) \to (0,0,0)$? Solution: use Stokes theorem. The curl of F is $[2y\cos(y^2), 2z, 3x^2]$. The surface $S : r(u,v) = [u,v,0]$ with $0 \leq u \leq 1$ and $0 \leq v \leq 1$ has C as boundary. Stokes allows to compute $\iint_S \text{curl}(F) \cdot dS$ instead. Since $r_u \times r_v = [0,0,1]$, the flux integral is $\int_0^1 \int_0^1 3u^2 dvdu = 1$. The computation of the line integral would have been more painful.

32.5. Problem: Compute the flux of the curl of $F(x,y,z) = [0,1,8z^2]^T$ through the upper half sphere S oriented outwards. **Solution:** Great, it is here, where we can use Stokes theorem $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$, where C is the boundary curve which can be parametrized by $r(t) = [\cos(t), \sin(t), 0]^T$ with $0 \leq t \leq 2\pi$. Before diving into the computation of the line integral, it is good to check, whether the vector field is a gradient field. Indeed, we see that $\text{curl}(F) = [0,0,0]$. This means that $F = \nabla f$ for some potential f implying by the **fundamental theorem of line integrals** that $\int_C F \cdot dr = 0$. But wait a minute, if the curl of F is zero, couldn't we just have seen directly that the flux of the curl through the surface is zero? Yes, we could have seen that before: for a gradient field, the flux of the curl of F through a surface is always zero, for the simple reason that the curl of such a field is zero.

32.6. Problem. What is the flux of the curl of $F(x,y,z) = [\sin(xyz), ze^{\cos(x+y)}, zx^5 + z^{22}]$ through the lower ellipsoid S given by $x^2/4 + y^2/9 + z^2/16 = 1, z < 0$? **Solution:** by Stokes theorem, it is the line integral $\int_C F \cdot dr$. Through the boundary $r(t) = [2\cos(t), 3\sin(t), 0]$. But in the xy-plane $z = 0$, the field F is zero. The result is zero.

32.7. Problem: What is the flux of the curl of F through an ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$? **Solution:** We can cut the ellipsoid into two parts to get two surfaces with boundary. The upper part $S_+ = \{(x,y,z) \in S, z > 0\}$ has the boundary $C_+ : r(t) = [2\cos(t), 3\sin(t), 0]$ which matches the orientation of the surface. Stokes theorem tells that $\iint_{S_+} \text{curl}(F) \cdot dS = \int_{C_+} F \cdot dr$. The lower part $S_- = \{(x,y,z) \in S, z < 0\}$ has the boundary $C_- : r(t) = [2\cos(t), -3\sin(t), 0]$ which matches the orientation of the lower part. Stokes theorem tells that $\iint_{S_-} \text{curl}(F) \cdot dS = \int_{C_-} F \cdot dr$. Together we have $\int_{C_-} F \cdot dr + \int_{C_+} F \cdot dr = 0$ as the line integrals have just different signs. The result is zero.

32.8. The left hand side of the **important formula** (it "imports" the curl) [2] is defined only in three dimensions. But the right hand side also makes sense **in** $\mathbb{R}^n$. It is $\text{tr}((dF)^*dr)$, where $*$ rotates the 2-frame by 90 degrees. The Stokes theorem for 2-surfaces works for $\mathbb{R}^n$ if $n \geq 2$. For $n = 2$, we have with $x(u,v) = u, y(u,v) = v$ the identity $\text{tr}((dF)^*dr) = Q_x - P_y$ which is Green's theorem. Stokes has the general structure $\boxed{\int_G \delta F = \int_{\delta G} F}$, where δF is a derivative of F and δG is the boundary of G.

Theorem: Stokes holds for fields F and 2-dimensional S in $\mathbb{R}^n$ for $n \geq 2$.

32.9. Why are we interested in $\mathbb{R}^n$ and not only in $\mathbb{R}^3$? One example is that 2-dimensional surfaces appear as "paths" which a moving string in 11 dimension traces. More important maybe is that statisticians work by definition in high dimensional spaces. When dealing with n data points, one works in $\mathbb{R}^n$. Why would you care about theorems like Stokes in statistics? As a matter of fact, integral theorems in general allow to **simplify computations**. As we have seen in Green's theorem, when computing the sum over all the curls, there are **cancellations** happening in the inside. Integral theorems "see these cancellations" and allow to **bypass and ignore stuff which does not matter**.

32.10. The fundamental theorem of line integrals $\int_a^b \text{tr}(df(r(t))dr(t))dt = f(r(b)) - f(r(a))$ holds also in $\mathbb{R}^n$. The flux integral

$$\iint_G \text{tr}(F^*(r(u,v))dr(u,v))\; dudv$$

is the analogue of a line integral in two dimensions. Written like this, we don't need the cross product. And not yet the language of **differential forms**.

32.11. Stokes deals with "fields" and "space". What happens if the field is space itself, that is if $F^* = dr$? It is of interest. For $m = 1$, and $F = dr^T$, then $\int_a^b |dr|^2\; dt$ is the **action integral** in physics. A general **Maupertius principle** assures that it is equivalent to the **arc length** $\int_a^b |dr|\; dt$ in the sense that minimizing arc length between two points is equivalent to minimize the action integral (which is more like the energy one uses to get from the first point to the second). Now, in two dimensions we have $\iint_G \text{tr}(dr^T dr)\; dudv$. We can compare this with $\iint_G \det(dr^T dr)\; dudv$ which is called the **Nambu-Goto action**, which resembles the **surface area** $\iint_G \sqrt{\det(dr^T dr)}\; dudv$ also called the **Polyakov action**. Nature likes to minimize. Free particles move on shortest paths, minimize the arc length. Maupertius tells that minimizing the length $\int_A^B |r'(t)|\; dt$ of a path equivalent to minimizing $\int_A^B r'(t) \cdot r'(t)\; dt$ which essentially is the integrated kinetic energy or gasoline use to go from A to B. For the purpose of minimizing stuff this also works for two dimensional actions. Minimizing the surface area $\iint_G |r_u \times r_v|\; dudv$ among all surfaces connecting two one dimensional curves is equivalent to minimize $\iint_G |r_u \times r_v|^2\; dudv$. Also in higher dimensions, Nambu-Goto and Polyakov are equivalent.

[2] I learned the "important formula" from Andrew Cotton-Clay in 2009:
http://www.math.harvard.edu/archive/21a_fall_09/exhibits/stokesgreen

Homework

Problem 32.1: Use Stokes to find $\int_C F \cdot dr$, where $F(x,y,z) = [12x^2y, 4x^3, 12xy + e^{(e^z)}]$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Problem 32.2: Evaluate the flux integral $\int\int_S \text{curl}(F) \cdot dS$, where

$$F(x,y,z) = [xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x]^T$$

and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 32.3: Find the line integral $\int_C F\ dr$, where C is the circle of radius 3 in the xz-plane oriented counter clockwise when looking from the point $(0,1,0)$ onto the plane and where F is the vector field

$$F(x,y,z) = [4x^2z + x^5, \cos(e^y), -4xz^2 + \sin(\sin(z))]^T .$$

Use a convenient surface S which has C as a boundary.

Problem 32.4: Find the flux integral $\int\int_S \text{curl}(F) \cdot dS$, where $F(x,y,z) =$

$$[2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz]^T$$

and S is the surface parametrized by

$$r(s,t) = [(1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s]^T$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.

Problem 32.5: Assume S is the surface $x^{22} + y^8 + z^6 = 100$ and $F = [e^{e^{22z}}, 22x^2yz, x - y - \sin(zx)]$. Explain why $\int\int_S \text{curl}(F) \cdot dS = 0$.

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