Unit 34: Stokes Applications

Topology

34.1. A region E in $\mathbb{R}^n$ is called **simply connected** if it is connected and for every closed loop C in E there is a continuous deformation C_s of C **within G** such that $C_0 = C$ and $C_1(t) = P$ is a point. For example, $C(t) = [\cos(t), \sin(t), 0]$ can be deformed in $E = \mathbb{R}^3$ to a point with $C_s(t) = [(1-s)\cos(t), (1-s)\sin(t), 0]$ as $C_1(t) = P = [0, 0, 0]$ for all t. Each Euclidean space $\mathbb{R}^n$ is simply connected. The region $G = \{x^2+y^2 > 0\} \subset \mathbb{R}^3$ is not simply connected as the circle $C : r(t) = [\cos(t), \sin(t), 0]$ winding around the z-axis can not be pulled together to a point **within** G. The region $G = \{x^2 + y^2 + z^2 > 0\} \subset \mathbb{R}^3$ is simply connected, but $G = \{x^2 + y^2 > 0\}$ in $\mathbb{R}^2$ is not. Remember that F was called **irrotational** if $\text{curl}(F) = 0$ everywhere.

Theorem: If F is irrotational on a simply connected E then $F = \nabla f$ in E.

34.2. Proof: since E is simply connected and $\text{curl}(F) = 0$, every closed loop C can be filled in by a surface $S = \bigcup_{0 \leq s \leq 1} C_s$ which has the boundary C. Stokes theorem gives $\int_S F \cdot dr = \iint_S \text{curl}(F) \cdot dS = 0$. The closed loop property implies path independence. A potential f can be obtained by fixing a base point p in E, then define for any other point x a path C_{px} going from p to x. The potential function f is then defined as $f(x) = \int_{C_{px}} F \cdot dr$. QED

34.3. The field $F(x, y, z) = [-y/(x^2 + y^2), x/(x^2 + y^2), 0]$ is defined everywhere except on the z-axis. The domain E, where F is defined is not simply connected. There is no global function f which is a potential for F.

34.4. The notion of "simply connectedness" is important in topology. The first solved **Millenium problem**, the **Poincaré conjecture**, is now a theorem. It tells that a 3-dimensional manifold which is simply connected is topologically equivalent to the 3-sphere $\{x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$. In two dimensions, the result was known for a long time already, because the structure of 2-dimensional connected manifolds is known.

Electromagnetism

34.5. The **Maxwell-Faraday equation** in electromagnetism relates the **electric field** E and the **magnetic field** B with the partial differential equation $\text{curl}(E) = -\frac{d}{dt}B$. Given a surface S, the flux integral $\iint_S B \cdot dS$ is called the **magnetic flux**

of B through the surface. If we integrate the Maxwell-Faraday equation, we see that $\iint_S \mathrm{curl}(E) \cdot dS$ is equal to minus the rate of change of the magnetic flux $-\frac{d}{dt} \iint_S B \cdot dS$. Stokes theorem now assures that $\iint_S \mathrm{curl}(E) \cdot dS = \int_C E \cdot dr$ is the line integral of the electric field along the boundary. But this is **electric potential** or voltage. We see:

We can generate an electric potential by changing the magnetic flux.

34.6. Changing the magnetic flux can happen in various ways. We can generate a changing magnetic field by using **alternating current**. This is how **transformers work**. An other way to change the flux is to **rotate a wire** in a fixed magnetic field. This is the **principle of the dynamo**:

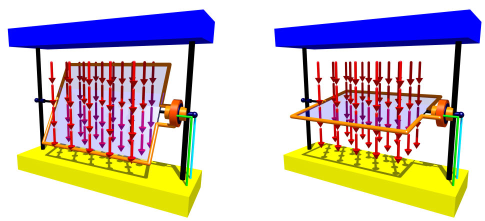

FIGURE 1. The dynamo, implemented using the ray tracer Povray. Electric current is generated by moving a wire in a fixed magnetic field.

34.7. The vector field $A(x, y, z) = \frac{[-y,x,0]}{(x^2+y^2+z^2)^{3/2}}$ is called the **vector potential** of a magnetic field $B = \mathrm{curl}(A)$. The picture shows some flow lines of this **magnetic dipole field** B. **Problem:** Find the flux of B through the lower half sphere $x^2 + y^2 + z^2 = 1, z \leq 0$ oriented downwards. **Solution:** Since we have an integral of the curl of the vector field A, we use **Stokes theorem** and integrate $A(r(t))$ along the boundary curve $r(t) = [\cos(t), -\sin(t), 0]$. First of all, we have $A(r(t)) = [\sin(t), \cos(t), 0]$. The velocity is $r\,'(t) = [-\sin(t), \cos(t), 0]$. The integral is $\int_0^{2\pi} -1 \; dt = -2\pi$.

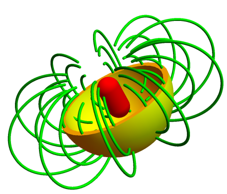

FIGURE 2. The flux of the magnetic field B through a surface can be computed with Stokes by computing a line integral of the vector potential A.

34.8. Here are all the four magical **Maxwell equations** for the **electric field** E and **magnetic field** B related to the **charge density** σ and the **electric current** j. The constant c is the speed of light. (By using suitable coordinates, one can assume $c = 1$.)

$$\boxed{\mathrm{div}(E) = 4\pi\sigma, \mathrm{div}(B) = 0, c \cdot \mathrm{curl}(E) = -B_t, c \cdot \mathrm{curl}(B) = E_t + 4\pi j}.$$

34.9. If F is the fluid velocity field and C is a closed curve, then $\int_C F \cdot dr$ is called the **circulation** of F along C. The curl of F is called the **vorticity** of F. A **vortex line** is a flow line of curl(F). Given a curve C, we can let any point in C flow along the vorticity field. This produces a **vortex tube** S. The flux of the vorticity though a surface S is the **vortex strength** of F through S. Stokes theorem implies the **Helmholtz theorem**.

Theorem: If C_s flows along F, then $\int_{C_s} F \cdot dr$ stays constant.

34.10. Proof: Let C be a closed curve and $C_s(t)$ be the curve after letting it flow using a deformation parameter s. The deformation produces a **tube surface** $S = \bigcup_{s=0}^{t} C_s$ which has the boundary C and C_t. Since the curl of F is always tangent to the surface S, the flux of the curl of F through S is zero. Stokes theorem implies that $\int_C F \cdot dr - \int_{C_s} F \cdot dr = 0$. The negative sign is because the orientation of C_s is different from the orientation of C if the surface has to be to the left.

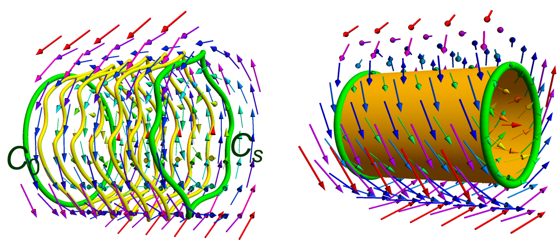

Figure 3. Helmholtz theorem assures that the circulation along a flux tube is constant. This is a direct application of Stokes theorem: because the curl of F is tangent to the tube, there is no flux through the tube.

Complex analysis

34.11. An application of Green's theorem is obtained, when integrating in the complex plane $\mathbb{C}$. Given a function $f(z) = u(z) + iv(z)$ from $\mathbb{C} \to \mathbb{C}$ and a closed path C parametrized by $r(t) = x(t) + iy(t)$ in $\mathbb{C}$, define the **complex integral** $\int_a^b (u(x(t) + iy(t)) + iv(x(t) + iy(t)))(x'(t) + iy'(t))\ dt$. This is $\int_a^b u(r(t))x'(t) - v(r(t))y'(t)\ dt + i\int_a^b v(r(t))x'(t) + u(r(t))'(t)\ dt$. These are two line integrals. The real part is $F = [u, -v]$, the imaginary part is $F = [v, u]$. Assume C bounds a region G, then Green's theorem tells that the first integral is $\iint_G -v_x - u_y dxdy$ and the second integral is $\iint_G u_x - v_y\ dxdy$. It turns out now that for nice functions f like polynomials, the **Cauchy-Riemann** differential equations $\boxed{u_x = v_y, v_x = -u_x}$ hold so that these line integrals are zero. We have therefore

Theorem: If f is a polynomial and C a closed loop, $\int_C f(z)\ dz = 0$

Homework: thanksgiving quickies

Problem 34.1: We can measure how many **magnetic monopoles** there are in the interior of a closed surface S by computing $\iint_S B \cdot dS$. We see that $B = \text{curl}(A)$ for a **magnetic potential** A, which is a vector field. What is $\iint_S B \cdot dS$? (We will see in the next lecture why this tells about the amount of magnetic monopoles inside S.)

Problem 34.2:
a) Define $\text{div}([P, Q, R]) = P_x + Q_y + R_z$. Check that $\text{div}(\text{curl}(F)) = 0$.
b) Is $\text{div}(\text{grad}(f)) = 0$ for all functions?
c) Is $\text{curl}(\text{curl}(F)) = [0, 0, 0]$ for all fields?
d) Which of the regions in Figure 4 are simply connected?
e) Which of the capital letters $A - Z$ are not simply connected?

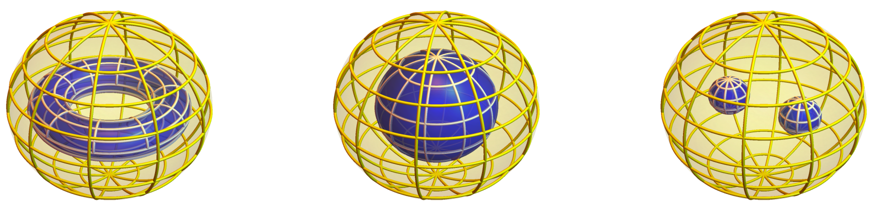

Figure 4. Complement $B \setminus T$ of the solid torus T in a ball B, the solid $\{1 < x^2 + y^2 + z^2 < 4\}$ or the complement of two small balls in a larger ball.

Problem 34.3: Let S be the torus $r(u, v) = [(3 + \cos(u))\cos(v), (3 + \cos(u))\sin(v), \sin(u)]$ and F the vector field $F(x, y, z) = [-y, x, 0]$. What is the flux of F through S? (No computation and no Stokes theorem is needed).

Problem 34.4: If F is a vector field, which is everywhere perpendicular to a surface S pointing in the normal direction of S, and $|F(x, y, z)| = 1$. What is $\iint_S F \cdot dS$?

Problem 34.5: a) Can you find a vector field F with $\text{curl}(F) = [0, x^2, 0]$?
b) Can you find a vector field F with $\text{curl}(F) = [0, 0, x^2]$?
c) Can you find a vector field $F = [P, Q, R]$ such that $\text{div}(F) = x^2$?
d) Can you find a gradient field $F = \nabla(f)$ such that $\text{div}(F) = x^2$?
e) Given a function $g(x, y, z)$, find F such that $\text{div}(F) = g$.