LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 35: Gauss theorem

Lecture

35.1. The **divergence** of a vector field $F = [P, Q, R]$ in $\mathbb{R}^3$ is defined as $\text{div}(F) = \nabla \cdot F = P_x + Q_y + R_z$. Let G be a solid in $\mathbb{R}^3$ bound by a surface S made of finitely many smooth surfaces, oriented so the normal vector to S points outwards. The **divergence theorem** or **Gauss theorem** is

Theorem: $\iiint_G \text{div}(F)\, dV = \iint_S F \cdot dS$.

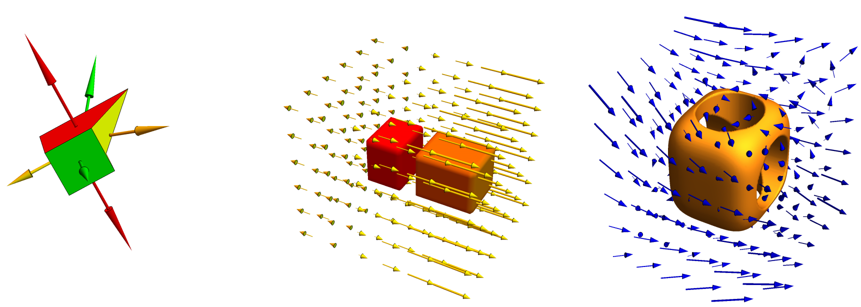

Figure 1. The boundary of a solid is oriented outwards. The divergence measures the expansion of a box flowing in the field. The flux of curl(F) through a closed surface is 0. No field is created inside.

35.2. Proof. If G is a solid of the form $G = \{(x,y,z) | (x,y) \in U, g(x,y) \leq z \leq h(x,y)\}$ and $F = [0,0,R]$, then $\iiint_G \text{div}(F)\; dV = \iint_U \int_{g(x,y)}^{h(x,y)} R_z\; dzdydx$ which is $\iint_G R(x,y,h(x,y)) - R(x,y,g(x,y))\; dydx$. The flux of $F = [0,0,R]$ through a surface $r(u,v) = [u,v,h(u,v)]$ is

$$\iint_G [0,0,R(u,v,h(u,v))] \cdot [-g_u, g_v, 1] dvdu = \iint_G R(x,y,h(x,y))\; dxdy\;.$$

Similarly, the flux through the bottom surface is $-\iint_G R(x,y,g(x,y))\; dxdy$. In general, write $F = [P,Q,R] = [P,0,0] + [0,Q,0] + [0,0,R]$ to get the claim for solid which are simultaneously bound by graphs of functions in x and y, or y and z or x and z. A general solid can be cut into such solids.

35.3. The theorem gives meaning to the term divergence. The total divergence over a small region is equal to the flux of the field through the boundary. If this is positive, then more field leaves than enters and field is "generated" inside. The divergence measures the expansion of the field. The field $F(x,y,z) = [x,0,0]$ for example expands, while $f(x,y,z) = [-x,0,0]$ compresses. $F(x,y,z) = [y,z,x]$ is "incompressible".

35.4. The divergence theorem holds in any dimension m. If $F = [F_1, \cdots, F_m]$ is the vector field, then $\partial_{x_1} F_1 + \cdot + \partial_{x_m} F_m$ is defined as the **divergence** of F. If G is an m-dimensional region with boundary $S = s(G)$, then the flux of F through S is defined as $\int_G F(s(u)) \cdot n(s(u)) |ds(u)|$, where $n(s(u))$ is a unit normal vector. This can be explained a bit better using the language of differential forms which is introduced next time.

35.5. The divergence of $F = [P,Q]$ is defined as $P_x + Q_y$. If $F^\perp = [Q,-P]$ is the turned vector field, then $\text{div}(F^\perp) = Q_x - P_y$ is the curl of F. Green's theorem tells that $\iint_G \text{curl}(F)\, dxdy$ which is $\iint_G \text{div}(F^\perp)\, dxdy$ is the line integral $\int_C F \cdot dr$. The line integral for F is the flux integral for $F^\perp$. The two dimensional divergence theorem is Green's theorem "turned".

Examples

35.6. Problem: Compute the flux of $F = [x,y,z]$ through the sphere of radius ρ bounding a ball G, oriented outwards. **Solution:** As $\text{div}(F) = 3$ we have $\iiint_G \text{div}(F) dV = 3\text{Vol}(G) = 3 \cdot 4\pi\rho^3/3$. The flux through the boundary is $\iint_S F \cdot dS$. As in spherical coordinates, $F(r(\phi,\theta)) \cdot r_\phi \times r_\theta = \rho^3 \sin(\phi)$, the flux is $\int_0^{2\pi} \int_0^\pi \rho^3 \sin(\phi)\, d\phi d\theta = 4\pi\rho^3$ also.

35.7. Problem: What is the flux of the vector field $F(x,y,z) = [6x + y^3, 3z^2 + 8y, 22z + \sin(x)]$ through the solid $G = [0,3] \times [0,3] \times [0,3] \setminus ([0,3] \times [1,2] \times [1,2] \cup [1,2] \times [0,3] \times [1,2] \cup [0,3] \times [0,3] \times [1,2])$ which is a cube with three perpendicular cubic holes which is the first stage of the **Menger sponge construction**? **Solution:** As $\text{div}(F) = 22 + 8 + 6 = 36$, the result is 36 times the volume of the solid which is $36(27-7) = 720$.

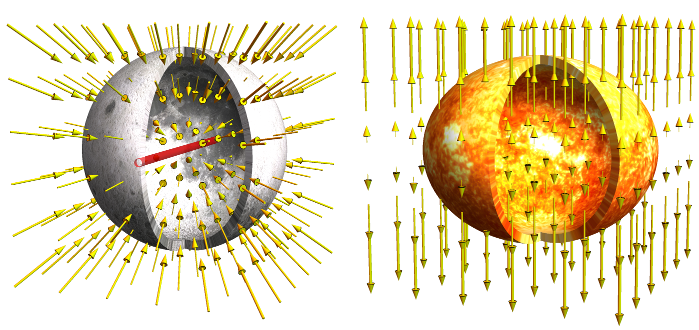

Figure 2. The gravity inside the moon is such that an elevator crossing the moon oscillates like a harmonic oscillator. The flux of $F = [0,0,z]$ through a surface is the volume inside.

35.8. Problem. How does the gravitational field look like inside the moon in distance ρ to the origin? **Solution.** A direct computation of summing up all the field values $F(x) = \iint_G (x-y)/|x-y|^3 dy$ is difficult as we can not compute in spherical coordinates. Fortunately we have the divergence theorem. The field $F(x)$ has constant length $F(\rho) = |F(x)|$ for x on a sphere $S(\rho)$ of radius ρ and points inwards. So $\iint_{S(\rho)} F \cdot dS = -4\pi\rho^2 F(\rho)$. Gauss was able to write down the gravitational field as a partial differential equation $\boxed{\operatorname{div}(F(x)) = 4\pi\sigma(x)}$, where $\sigma(x)$ is the mass density of the solid. We see then with the divergence theorem that $\iiint_{B(\rho)} 4\pi\sigma(x)\, dx$ is equal to $-4\pi\rho^2 F*(\rho)$. Assuming σ to be constant, we have $4\pi(4\pi\rho^3/3)\sigma = -4\pi\rho^2 F(\rho)$ which gives $F(\rho) = (4\sigma/3)\rho$. The field grows linearly inside the body. If ρ is bigger than the radius of the moon, then $\iiint_{B(\rho)} 4\pi\sigma(x)\, dx$ is $4\pi M$, where $M = \iiint_G \sigma(x)\, dx$ is the mass of the moon. We see that in that case $F(\rho) = M/\rho^2$, which is the Newton law.

35.9. Problem: Compute using the divergence theorem the flux of the vector field $F(x,y,z) = [2342434y, 2xy, 4yz + 21341324xy]^T$ through the unit cube $[0,1] \times [0,1] \times [0,1]$ which is opened on the top. **Solution:** the divergence of F is $2x + 4y$. Integrating this over the unit cube gives $1 + 2 = 3$. The flux through all 6 faces is 3. The flux through the face $z = 1$ is $\int_0^1 \int_0^1 4y\, dxdy = 2$. We have to subtract this and get $3 - 2 = 1$.

35.10. Similarly as Green's theorem allowed area computation using line integrals the volume of a region can be computed as a flux integral: take a vector field F with constant divergence 1 like $F(x,y,z) = [0,0,z]$. We have $\int\int_S [0,0,z] \cdot dS = \text{Vol}(G)$.

35.11. Example: For an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2$, where the parametrization is $r(\phi,\theta) = [a\sin(\phi)\cos(\theta), b\sin(\phi)\sin(\theta), c\cos(\phi)]$, we have $[0,0,c\cos(\phi)][ab\sin(\phi)\cos(\phi)] = abc\sin(\phi)\cos^2(\phi)$ leading to $2\pi abc2/3 = 4\pi abc/3$.

35.12. A computer can determine the volume of a solid enclosed by a triangulated surface by computing the flux of the vector field $F = [0,0,z]$ through the surface. The vector field has divergence 1 so that by the divergence theorem, the flux gives the volume. A computer stores a geometric object using triangles. Assume ABC is that triangle. If $n = AB \times AC$ points outside the region, then the flux is $F \cdot n/2$. A computer can now add up all these values and get the volume.

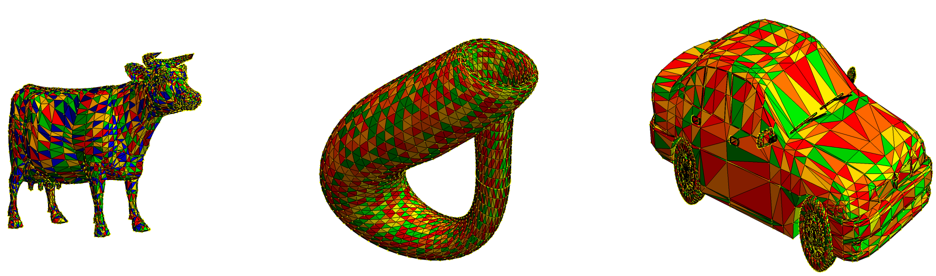

FIGURE 3. A cow, a Klein bottle and a car from the Mathematica example files and produce closed surfaces. The Klein bottle does not have an interior however.

Homework

Problem 35.1: Use the divergence theorem to calculate the flux of $F(x,y,z) = [x^3, y^3, z^3]^T$ through the sphere $S : x^2 + y^2 + z^2 = 1$, where the sphere is oriented so that the normal vector points outwards.

Problem 35.2: Assume the vector field

$$F(x,y,z) = [5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z)]^T$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\iint_S F \cdot dS$.

Problem 35.3: Find the flux of the vector field $F(x,y,z) = [xy, yz, zx]^T$ through the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$.

Problem 35.4: Find the flux of $F(x,y,z) = [x+y+z, x+z, z+y]^T$ through the Menger sponge M_n defined in the unit cube and take the limit $n \to \infty$.

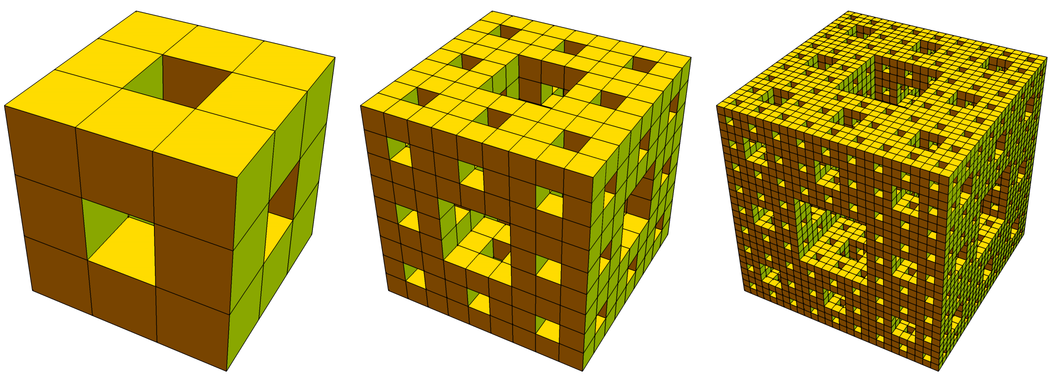

Figure 4. Approximations to the Menger sponge.

Problem 35.5: Compute the flux of the vector field $F(x,y,z,w) = [x + 2y^2, 3x + 4y^5, 6z + 8z^9, 7w + 9x^{10}]^T$ through the three 3-sphere $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$, oriented outwards.

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