

# LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

## Unit 36: General Stokes

### LECTURE

**36.1.** We know already  $E = \mathbb{R}^n = M(n, 1)$ , the space of **column vectors** and its **dual**  $E^* = M(1, n)$ , the space of **row vectors**. To get more general objects, it is important to think about vectors as **maps**. A row vector is a linear map  $F : E \rightarrow \mathbb{R}$  defined by  $F(u) = Fu$  and a column vector defines a linear map  $F : E^* \rightarrow \mathbb{R}$  by  $F(u) = uF$ . A map  $F(x_1, \dots, x_n)$  of several variables is called **multi-linear**, if it is linear in each coordinate. The set  $T_q^p(E)$  of all **multi-linear maps**  $F : (E^*)^p \times E^q \rightarrow \mathbb{R}$  is the space of **tensors of type**  $(p, q)$ . We have  $T_0^1(E) = E$  and  $T_1^0(E) = E^*$  and  $T_1^1(E)$  is  $M(n, n)$  the space of  $n \times n$  matrices: given a matrix  $A$ , a column vector  $v \in E$  and a row vector  $w \in E^*$ , we have the bi-linear map  $F(v, w) = wAv$ . It is linear in  $v$  and in  $w$ .

**36.2.** Let  $\Lambda^q(E)$  be the subspace of  $T_q^0(E)$  which consists of tensors  $F$  of type  $(0, q)$  such that  $F(x_1, \dots, x_q)$  is anti-symmetric in  $x_1, \dots, x_q \in E$ : this means  $F(x_{\sigma(1)}, \dots, x_{\sigma(q)}) = (-1)^\sigma F(x_1, \dots, x_q)$  for all  $i, j = 1, \dots, q$ , where  $(-1)^\sigma$  is the **sign** of the permutation  $\sigma$  of  $\{1, \dots, n\}$ . If the **Binomial coefficient**  $B(n, q) = n!/(q!(n-q)!)$  counts the number of subsets with  $q$  elements  $i_1 < \dots < i_q$  of  $\{1, \dots, n\}$  and  $E$  has dimension  $n$ , then  $\Lambda^q(E)$  has dimension  $B(n, q)$ . A map  $F : E \rightarrow T_q^p(E)$  is called a  $(p, q)$ -**tensor field**. The set  $T_0^1(E)$  is the space of **vector fields**. If  $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a smooth map, then  $F = d^k g$  is a tensor field of type  $(0, k)$ . A  **$k$ -form** is a  $(0, k)$ -tensor field  $F$  with  $F(x) \in \Lambda^k(E)$ . A 2-form in  $\mathbb{R}^3$  for example attaches to  $x \in \mathbb{R}^3$  a bi-linear, anti-symmetric map  $F(x)(u, v) = -F(x)(v, u)$ . One writes  $Pdydz + Qdxdz + Rdx dy$  where  $dydz(u, v) = u_2v_3 - u_3v_2$ ,  $dxdz(u, v) = u_1v_3 - u_3v_1$ ,  $dx dy(u, v) = u_1v_2 - v_1u_2$ .

**36.3.** The **exterior derivative**  $d : \Lambda^p \rightarrow \Lambda^{p+1}$  is defined for  $f \in \Lambda^0$  as  $df = f_{x_1} dx_1 + \dots + f_{x_n} dx_n$  and  $d(f dx_{i_1} \dots dx_{i_p}) = \sum_i f_{x_i} dx_i dx_{i_1} \dots dx_{i_p}$ . For  $F = Pdx + Qdy$  for example, it is  $(P_x dx + P_y dy)dx + (Q_x dx + Q_y dy)dy = (Q_x - P_y) dx dy$  which is the **curl** of  $F$ . If  $r : G \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a parametrization, then  $S = r(G)$  is a  **$m$ -surface** and  $\delta S = r(\delta G)$  is its **boundary** in  $\mathbb{R}^n$ . If  $F \in \Lambda^p(\mathbb{R}^n)$  is a  **$p$ -form** on  $\mathbb{R}^n$ , then  $r^*F(x)(u_1, \dots, u_p) = F(r(x))(dr(x)(u_1), dr(x)(u_2), \dots, dr(x)(u_p))$  is a  $p$ -form in  $\mathbb{R}^m$  called the **pull-back** of  $r$ . Given a  $p$ -form  $F$  and an  $p$ -surface  $S = r(G)$ , define the integral  $\int_S F = \int_G r^*F$ . The **general Stokes theorem** is

**Theorem:**  $\int_S dF = \int_{\delta S} F$  for a  $(m-1)$ -form  $F$  and  $m$  surface  $S$  in  $E$ .

**36.4.** Proof. As in the proof of the divergence theorem, we can assume that the region  $G$  is simultaneously of the form  $g_j(x_1, \dots, \hat{x}_j, \dots, x_m) \leq x_j \leq h_j(x_1, \dots, \hat{x}_j, \dots, x_m)$ , where  $1 \leq j \leq n$  and that  $F = [0, \dots, 0, F_j, 0, \dots, 0]$ . The coordinate independent definition of  $dF$  reduces the result to the divergence theorem in  $G$ . QED

### EXAMPLES

**36.5.** For  $n = 1$ , there are only 0-forms and 1-forms. Both are scalar functions. We write  $f$  for a 0-form and  $F = f dx$  for a 1-form. The symbol  $dx$  abbreviates the linear map  $dx(u) = u$ . The 1-form assigns to every point the linear map  $f(x)dx(u) = f(x)u$ . The exterior derivative  $d : \Lambda^0 \rightarrow \Lambda^1$  is given by  $df(x)u = f'(x)u$ . Stokes theorem is the **fundamental theorem of calculus**  $\int_a^b f'(x)dx = f(b) - f(a)$ .

**36.6.** For  $n = 2$ , there are 0-forms, 1-forms and 2-forms. It is custom to write  $F = Pdx + Qdy$  rather than  $F = [P, Q]$  which is thought of as a linear map  $F(x, y)(u) = P(x, y)u_1 + Q(x, y)u_2$ . A 2-form is also written as  $F = f dx dy$  or  $F = f dx \wedge dy$ . Here  $dx dy$  means the bi-linear map  $dx dy(u, v) = (u_1 v_2 - u_2 v_1)$ . The 2-form defines such a bi-linear map at every point  $(x, y)$ . The exterior derivative  $d\Lambda^0 \rightarrow \Lambda^1$  is  $df(x, y)(u_1, u_2) = f_x(x, y)u_1 + f_y(x, y)u_2$  which encodes the Jacobian  $df = [f_x, f_y]$ , a row vector. The exterior derivative of a 1-form  $F = Pdx + Qdy$  is  $dF(x, y)(u, v) = (-1)^1 P_y(x, y) \det([u, v]) + (-1)^2 Q_x(x, y) \det([u, v])$  which is  $(Q_x - P_y) dx dy$ . Using coordinates is convenient as  $dF = P_y dy dx + Q_x dx dy = (Q_x - P_y) dx dy$  using now that  $dy dx = -dx dy$ .

**36.7.** For  $n = 3$ , we write  $F = Pdx + Qdy + Rdz$  for a 1-form, and  $F = Pdydz + Qdzdx + Rdx dy$  for a 2-form. Here  $dy dz = dy \wedge dz$  are symbols representing bi-linear maps like  $dy dz(u, v) = u_2 v_3 - v_3 u_2$ . As a 2-form has 3 components, it can be visualized as vector field. A 3-form  $f dx dy dz$  defines a scalar function  $f$ . The symbol  $dx dy dz = dx \wedge dy \wedge dz$  represents the map  $dx dy dz(u, v, w) = \det([uvw])$ . The exterior derivative of a 1-form gives the curl because  $d(Pdx + Qdy + Rdz) = P_y dy dx + P_z dz dx + Q_x dx dy + Q_z dz dy + R_x dx dz + R_y dy dz$  which is  $(R_y - Q_z) dy dz + (P_z - R_x) dz dx + (Q_x - P_y) dx dy$ . The exterior derivative of a 2-form  $Pdydz + Qdzdx + Rdx dy$  is  $P_x dx dy dz + Q_y dy dz dx + R_z dz dx dy = (P_x + Q_y + R_z) dx dy dz$ . To integrate a 2-form  $F = x^2 y z dx dy + y z dy dz + x z dx dz$  over a surface  $r(u, v) = [x, y, z] = [uv, u - v, u + v]$  with  $G = \{u^2 + v^2 \leq 1\}$  we end up with integrating  $F(r(u, v)) \cdot r_u \times r_v$ . In order to integrate  $dF$  for a 1-form  $F = Pdx + Qdy + Rdz$  we can also pull back  $F$  and get  $\iint_G F_v(r(u, v)) r_u - F_u(r(u, v)) r_v \, du dv$ .

**36.8.** For  $n = 4$ , where we have 0-forms  $f$ , 1-forms  $F = Pdx + Qdy + Rdz + Sdw$  and 2-forms  $F = F_{12} dx dy + F_{13} dx dz + F_{14} dx dw + F_{23} dy dz + F_{24} dy dw + F_{34} dz dw$  which are objects with 6 components. Then 3-forms  $F = Pdy dz dw + Qdx dz dw + Rdx dy dw + Sdx dy dz$  and finally 4-forms  $f dx dy dz dw$ .

### REMARKS

**36.9.** Historically, differential forms emerged in 1922 with Élie Cartan. Most textbooks introduce the Grassmanian algebra early and use the language of “chains” for example which is the language used in algebraic topology. It was Jean Dieudonné in 1972 who freed the general Stokes theorem from chains and used first the coordinate free pull

back idea. This allowed us in this lecture to formulate the general Stokes theorem from scratch **on a single page** with all definitions.

**36.10. What is a differential form?** We have seen a mathematically precise definition: a differential form is a **kind of field**: a multi-linear anti-symmetric function attached to each point of space. But what is the intuition and what are ways to “visualize” and “see” and “understand” such an object? Here are four paths. Maybe one of them helps:

**A)** Using **Stokes** one can see a form as a functional  $F$ , which assigns to a  $m$ -dimensional oriented surface  $S$  a number  $\int_S F \cdot dS$  such that  $\int_{-S} F \cdot dS = \int_S (-F) \cdot dS = -\int_S F \cdot dS$ .<sup>1</sup> This way of thinking about forms matches what we do in the discrete. If we have a  $k$ -form on a graph, then this is a function on  $k$ -dimensional oriented complete subgraphs. Given a graph  $S$  we have  $\int_S F \cdot dS = \sum_{x \in S} F(x)$ , where the sum is over all  $k$ -dimensional simplices in  $S$ .

**B)** One can understand differential forms better using arithmetic, the **Grassmanian algebra**. This is done with the help of the **tensor product**, which induces an **exterior product**  $F \wedge G$  on  $\Lambda^p \times \Lambda^q \rightarrow \Lambda^{p+q}$ . This product generalizes the cross product  $\Lambda^1 \times \Lambda^1 \rightarrow \Lambda^2$  which works for  $n = 3$  as there, the space of 1-forms  $\Lambda^1$  and 2-forms  $\Lambda^2$  can be identified. The exterior algebra structure helps to understand  $k$ -forms. We can for example see a 2-form as an exterior product  $F \wedge G$  of two 1-forms. We can think of a 2-form for example as attaching two vectors at a point and identify two such frames if their orientation and parallelogram areas match.

**C)** A third way comes through **physics**. We are familiar with manifestations of electromagnetism: we see light, we use magnets to attach papers to the fridge or have magnetic forces keep the laptop lid closed. Electric fields are felt when combing the hair, as we see sparks generated by the high electric field obtained by stripping away the electrons from the head. We use magnetic fields to store information on hard drives and electric fields to store information on a SSD harddrive. Non-visible electromagnetic fields are used when communicating using cell phones or connecting through blue-tooth or wireless network connections. The electro-magnetic field  $E, B$  is actually a 2-form in 4-dimensions. The  $B(4, 2) = 6$  components are  $(E_1, E_2, E_3, B_1, B_2, B_3)$ .

**D)** A fourth way comes through **discretization**. When formulating Stokes on a discrete network, everything is much easier: a  $k$ -form is just a function on oriented  $k$ -dimensional complete subgraphs of a network. Start with a graph  $G = (V, E)$  and orient the complete subgraphs arbitrarily. Given a  $k$ -form  $F$ , a function on  $k$ -simplices has an exterior derivative at a  $k + 1$  dimensional simplex  $x$  is defined as  $dF(x) = \sum_{y \subset x} \sigma(y, x) F(y)$ , where the sum is over all  $k$ -dimensional sub-simplices of  $x$  and  $\sigma(y, x) = 1$  if the orientation of  $y$  matches the orientation of  $x$  or  $-1$  else. We have for example seen that for a 1-form  $F$ , a function on edges, the exterior derivative at a triangle  $x$  is the sum over the  $F$  values of the edges, where we add up the value negatively if the arrow of the edge does not match the orientation of the triangle.

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<sup>1</sup>David Bachman’s text on differential forms: “it is a thing which can be integrated”.

## APPLICATIONS

**36.11.** An **electromagnetic field** is determined by a 1-form  $A$  in 4-dimensional space time. The electromagnetic field is  $F = dA$ . The Maxwell equations are  $dF = 0$  (the relation  $d \circ d = 0$  is seen in the homework). The second part of the Maxwell equations are  $d^*F = j$ , where  $d^* : \Lambda^p \rightarrow \Lambda^{p-1}$  is the adjoint and  $j$  is a 1-form encoding both the electric charge and the electric current. We can always gauge with a gradient  $A + df$  so that  $d^*(A + df) = 0$ . The Maxwell equations reduced to the Poisson equation  $LF = (dd^* + d^*d)F = j$ , where  $L$  is the **Laplacian** on 1-forms. In vacuum, without electric charges or currents, we have the **wave equation**  $LF = 0$ . And there was light.

## HOMEWORK

**Problem 36.1:** Given the 1-form  $F(x, y, z, w) = [x^3, y^5, z^5, w^2] = x^3dx + y^5dy + z^5dz + w^2dw$  and the curve  $C : r(t) = [\cos(t), \sin(t), \cos(t), \sin(t)]$  with  $0 \leq t \leq \pi$ . Find the line integral  $\int_C F(r(t)) \cdot dr$ .

**Problem 36.2:** Given the 1-form  $F = [xyz, xy, wx, wxy] = xyzdx + xydy + wxdz + wxydw$ , find the curl  $dF$ . Now find  $\iint_S dF$  over the 2-dimensional surface  $S : x^2 + y^2 \leq 1, z = 1, w = 1$  which has as a boundary the curve  $C : r(t) = [\cos(t), \sin(t), 1, 1]^T, 0 \leq t \leq 2\pi$ . You certainly can use the Stokes theorem. If you like to compute both sides of the theorem you can see how the theorem works. The 2-manifold  $S$  is parametrized by  $r(t, s) = [s, t, 1, 1]^T$ . The  $(r_s \wedge r_t)_{ij}$  has 6 components, where only one component  $(r_s \wedge r_t)_{12}$  is nonzero. This will match with the  $dF_{12} = Pdx dy$  part of the 6-component 2-form  $dF$  building the curl. We will have to integrate then over  $G = s^2 + t^2 \leq 1$ .

**Problem 36.3:** Given the 2-form  $F = z^4xdxdz + xyzw^2dydw$  and the 3-sphere  $x^2 + y^2 + z^2 + w^2 = 1$  oriented outwards. What is the integral  $\iiint_S dF$ ? To compute this 3D integral, you can use the general integral theorem.

**Problem 36.4:** Given the 3-form  $F = xyzdxdydz + y^2zdydzdw$ , find the divergence  $dF$ . Now find the flux of  $F$  through the unit sphere  $x^2 + y^2 + z^2 + w^2 = 1$  oriented outwards.

**Problem 36.5:**

- Take  $f(x, y, z, w)$ . Check that  $F = df$  satisfies  $dF = 0$ .
- Take  $F = F_1dx + F_2dy + F_3dz + F_4dw$ . Compute the curl  $G = dF$  and check that  $dG = 0$ .
- Take the 2-form  $F = F_{12}dxdy + F_{13}dxdz + F_{14}dxdw + F_{23}dydz + F_{24}dydw + F_{34}dzdw$ . Write down the 3-form  $G = dF$  and check  $dG = 0$ .
- Take the 3-form  $F = F_1dydzdw + F_2dxdzdw + F_3dxdydw + F_4dxdydz$  and compute the 4-form  $G = dF$ . Check that  $dG = 0$ .