Unit 37: A Discrete World

Seminar

37.1. A 0**-form** f on a graph $G = (V, E)$ is a function on the vertices V. It is what we call a **scalar function**. A 1**-form** is a function on the oriented edges E meaning $F(a, b) = -F(b, a)$. Informally, as in the continuum, we think of a 1-form as a **vector field**. The **gradient** $F(a, b) = df(a, b) = f(b) - f(a)$ of a 0-form f is a 1 form F. The **curl** of a vector field F is a 2**-form**. It is a function on triangles (a, b, c) given by $dF(a, b, c) = F(a, b) + F(b, c) + F(c, a)$ which can be seen as the line integral along the boundary of the triangle. When describing p-forms for $p > 0$, orientation matters. To fix it, just enumerate the vertices V and then choose the orientation of an edge (a, b) with $a < b$ or the orientation of a triangle (a, b, c) if $a < b < c$. The discrete **Stokes theorem** $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$ told us that that the sum of the curls of F on triangles of a surface S is equal to the line integral of F along the boundary C of S.

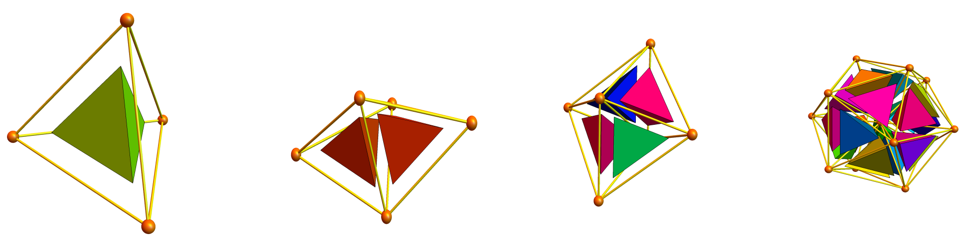

FIGURE 1. Examples of three dimensional graphs with 1,2,4 and 12 tetrahedra. The divergence $dF(x)$ of a 2-form F is the sum over all values $F(y)$, where $y \subset x$ runs over the triangular faces of x. The sum of all divergences is the flux of F through the boundary because in the inside the fluxes cancel. This is the divergence theorem for solids.

37.2. A **tetrahedral graph** is a collection of 4 nodes which all are connected to each other. A 3**-form** on a graph G is a function on tetrahedral sub-graphs x of G. An example is the **divergence** $dF(x)$ **of a 2-form** F which is defined as the sum of the $F(y)$ values of the triangles $y \subset x$ enclosing the tetrahedron x. As in the continuum, the orientation plays a role. Here is the **discrete divergence theorem** for a solid G is built by tetrahedra x and where the boundary surface S consists of triangles:

Problem A: Check that $\sum_{x \in G} \text{div}(F)(x) = \sum_{y \in S} F(y)$.

Hint: prove by induction with respect to the number of tetrahedra. first check that if G is a single tetrahedron, this is the definition of the divergence. Then see what happens if a new tetrahedron is added.

37.3. We also have seen that the divergence of the curl of a vector field F is zero: We had $\text{curl}(F) = [R_y - Q_z, P_z - R_x, Q_x - P_y]$ and taking the x derivative of $R_y - Q_z$ is $R_{yx} - Q_{zx}$, the y derivative of $P_z - R_x$ is $P_{zy} - R_{xy}$ and the z-derivative of $Q_x - P_y$ is $Q_{xz} - P_{yz}$. Adding them all up gives 0. In the discrete it is even simpler. Start with a 1-form F on the edges of a graph. Then form the curls, which are functions on the triangles, then add up all these curls. You check:

Problem B: Check: $\text{div}(\text{curl}(F))(x) = 0$ of every F and tetrahedron x.

37.4. The general Stokes theorem is not much different. A **p-simplex** in a graph is a collection of $p+1$ nodes which are all connected to each other. A **p-form** is a function on the set of p-simplices x in G. The function value is fixed if the simplex is given in an oriented way but defined also if the simplices are oriented differently, we just have $F(x_0,\dots,x_p) = (-1)^\sigma f(\sigma(x_0),\dots,\sigma(x_p))$ if σ is a permutation. For example $F(x_0,x_1,x_2) = F(x_1,x_2,x_0) = F(x_2,x_0,x_1) = -F(x_1,x_0,x_2) = -F(x_0,x_2,x_1) = -F(x_2,x_1,x_0)$.

37.5. The **exterior derivative** of p-form F is the $(p+1)$-form

$$dF(x_0,\dots,x_{p+1}) = \sum_{j=0}^{p+1} (-1)^j F(x_0,\dots,\hat{x_j},\dots x_{p+1}) \; .$$

Problem C: Check in general that $ddF = 0$.

37.6. The general Stokes theorem tells that for a m-dimensional graph G with boundary S and a $(m-1)$-form F we have

Theorem: $\sum_{x \in G} dF(x) = \sum_{y \in S} F(y)$

Gravity

37.7. The **Newton equations** $\frac{d^2}{dt^2} x_k = -\sum_j G m_j / |x_k - x_j|^2$ with gravitational constant G describe the motion of finitely many mass points with positions $x_k(t) \in \mathbb{R}^3$ and mass m_k. These classical laws govern the motion of **planets** in our solar system, **stars** in a galaxy or **galaxies** in a **galaxy cluster**. While relativity modifies this Newtonian picture slightly and produces corrections which for example manifest in the Perihel advancement of Mercury, the Newtonian theory is amazingly accurate. Gauss derived the gravitational inverse square force F from $\text{div}(F) = 4\pi\sigma$, where σ is the mass density. While divergence usually maps a 2-form to a 3-form, it is the adjoint d^* of the gradient d. In $\mathbb{R}^3$ it is equivalent. Now, $L = \text{div} \circ \text{grad} = d^* d : \Lambda^0 \to \Lambda^0$ is called the **Kirchhoff Laplacian**. The Gauss law of gravity therefore is the **Poisson equation** $\boxed{LV = 4\pi\sigma}$, where V is the gravitational potential, a 0−form. Since $d^* = 0$ on 0-forms, we can also write $L = dd^* + d^* d$. Classical gravity gets from a mass density σ the gravitational potential V and so the gravitational field as a gradient $F = dV$:

$(d^*d + dd^*)V = 4\pi\sigma$ defines the gravitational 1-form $F = dV$.

Electromagnetism

37.8. The **Maxwell equations** $\mathrm{div}(E) = 4\pi\sigma$, $\mathrm{div}(B) = 0$, $\mathrm{curl}(E) = -B_t$, $\mathrm{curl}(B) = E_t + 4\pi i$ become more elegant when written in four-dimensional **space-time** $\mathbb{R}^4$. There are then two equations only. The first is $dF = 0$ which is evident from $F = dA$ and $d^2 = 0$. The second is $d^*F = 4\pi j$, where j is the 4-**current** encoding both the charge density σ as well as the electric current i. Now $dF = 0$ implies in a simply connected region that $F = dA$, where A is an **electro-magnetic potential**. If $d^*A = 0$ (which can always be achieved by adding a gradient to A) we get the Poisson equation $LA = (dd^* + d^*d)A = 4\pi j$. This completely encodes the Maxwell equations; we can look at it also in a discrete network. Classical electromagnetism in a world with charge and current density j is the field $F = dA$, where A is obtained from

$(d^*d + dd^*)A = 4\pi j$ defines the electromagnetic 2-form $F = dA$.

Quantum mechanics and beyond

37.9. In this last homework we deal with a small universe G. We call it **Gaia**, the primordial deity of earth. In Greek mythology, Gaia was the daughter of **Aether** the god of air and **Hemera** the goddess of light. We only create the gravitational field, the electromagnetic field on G and some quanta, so there will be matter and light in this world. But that mathematics is exactly as in the universe we live in: the classical gravitational field is described with the language of Gauss which we have seen to imply the Newton law of gravity. The electromagnetic field is formulated according to Maxwell, but directly in space-time. We also look a bit at quantum mechanics as the eigenvalues and eigenvectors of the Laplacian L play a role when looking at the **Wheeler De Witt** equation, a time-independent Schrödinger equation in space time

$(d^*d + dd^*)F = \lambda F$ defines a wave function F on p-forms.

37.10. The rest will be up to you: it remains to include the Fermionic constituents of matter (quarks (building mesons and baryons) as well as leptons) and bosons (photons, gluons, vector bosons and the Higgs) as well as a few other details. Don't worry, a former student has solved a similar homework assignment in less than 7 days ...

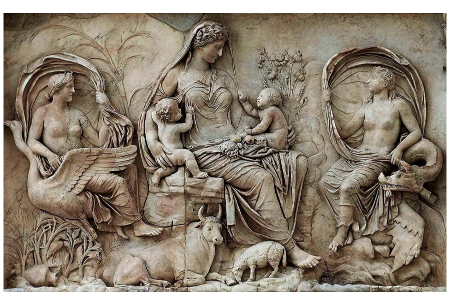

Figure 2. The Greek godess Gaia, seen in a Roman relief sculpture from the "Ara Pacis Augustae" in Rome. (Image by Dr. Sarah E. Bond.)

Homework

Problem 37.1: Given the 1-form F in Figure 3a, find the 0-form $f(x) = d^*F(x) = \sum_{e,e\to x} F(e)$. Check $\sum_{x\in V} d^*F(x) = 0$. (Conservation law)

Problem 37.2: a) Given the 0-form f in Figure 3b, find $F = df$, then compute $d^*F = d^*df = Lf$.
b) Given the 2-form H in Figure 3c, find a 1 form F such that $dF = H$.

Problem 37.3: Given the 0-form f in Figure 4a check that this is f satisfies $L_0 f = \lambda f$ for some constant λ. This is called an eigenvalue of L. We write L_p for $dd^* + d^*d$ restricted to p-forms.

Problem 37.4: In Figure 4c you see a 2-form H. Check that this is H satisfies $L_2 H = \lambda H$ for some constant λ. What is the eigenvalue?

Problem 37.5: Find $f = d^*F$ and $H = dF$ for the 1-form F in Figure 4b. Then check $d^*H + df = (d^*d + dd^*)F = \lambda F$ for some constant λ.

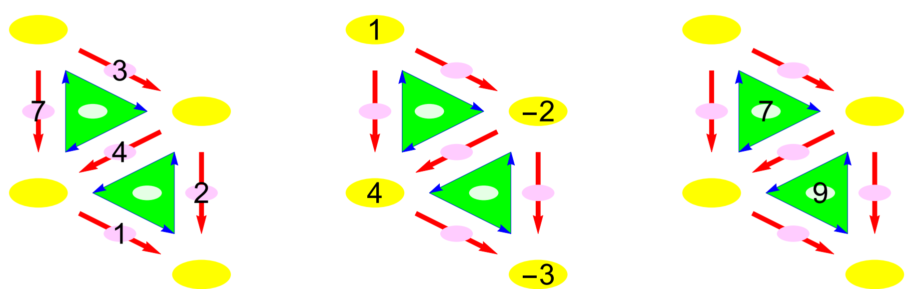

Figure 3. a) a 1-form,b) a 0-form, and c) a 2-form

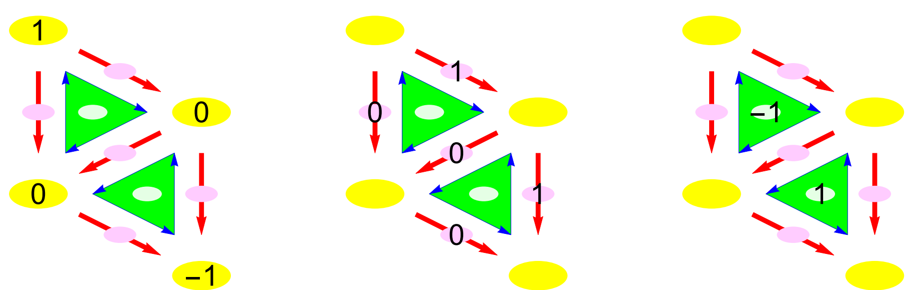

Figure 4. Eigenvectors. a) for L_0 b) for L_1 and c) for L_2.

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