LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

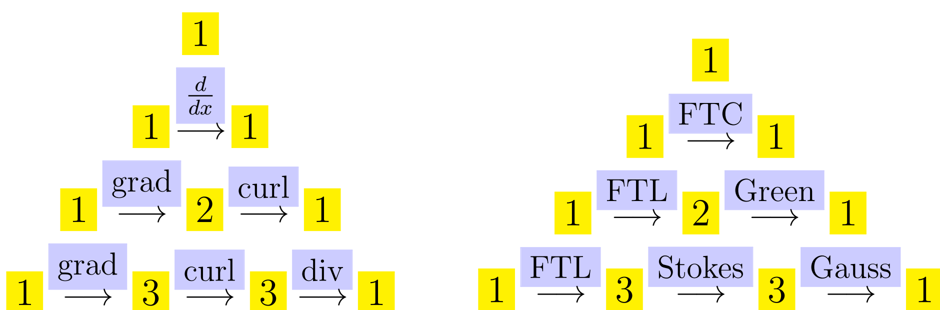
Unit 38: Geometries and Fields

LECTURE

38.1. Integral theorems deal with **geometries** G and **fields** F. **Integration** pairs them up and gives the **Stokes theorem**

$$\int_G dF = \int_{\delta G} F$$

It involves the **boundary** δG of G and the **exterior derivative** dF of F. One can classify the theorems by looking at the dimension n of space and the dimension m of the object we are integrating over. In dimension n, there are n theorems:



38.2. The **Fundamental theorem of line integrals** is a theorem about the gradient ∇f. It tells that if C is a curve going from A to B and f is a function (that is a 0-form), then

Theorem: $\int_C \nabla f \cdot dr = f(B) - f(A)$

In calculus we write the 1-form as a column vector field ∇f. It actually is a 1-form $F = df$, a field which attaches a row vector to every point. If the 1-form is evaluated at $r'(t)$ one gets $df(r(t))(r'(t))$ which is the matrix product. We integrate then the **pull back** of the 1-form on the interval $[a, b]$. It is the switch from row vectors to column vectors which leads to the **dot product** $\nabla f(r(t)) \cdot r'(t)$. For closed curves, the line integral is zero. It follows also that integration is **path independent**.

38.3. Green's theorem tells that if $G \subset \mathbb{R}^2$ is a region bound by a curve C having G to the left, then

Theorem: $\iint_G \text{curl}(F)\, dxdy = \int_C F \cdot dr$

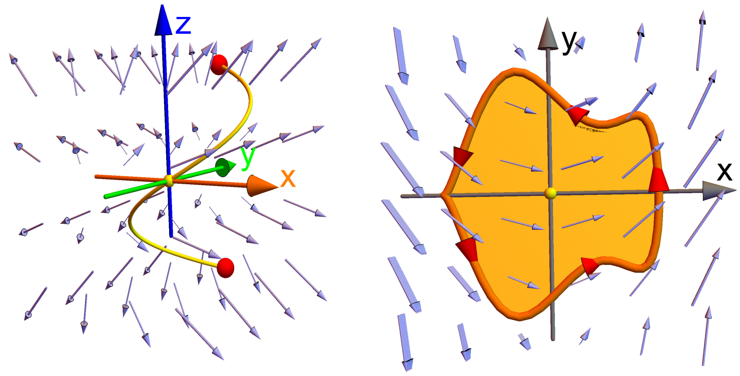


Figure 1. Fundamental theorem of line integrals and Green's theorem.

In the language of forms, $F = Pdx + Qdy$ is a 1-form and $dF = (P_x dx + P_y dy)dx + (Q_x dx + Q_y dy)dy = (Q_x - P_y)dxdy$ is a 2-form. We write this 2-form dF as $Q_x - P_y$ and treat it as a scalar function even so this is not the same as a 0-form, which is a scalar function. If $\text{curl}(F) = 0$ everywhere in $\mathbb{R}^2$ then F is a gradient field.

38.4. Stokes theorem tells that if S is a surface with boundary C oriented to have S to the left and F is a vector field, then

Theorem: $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$

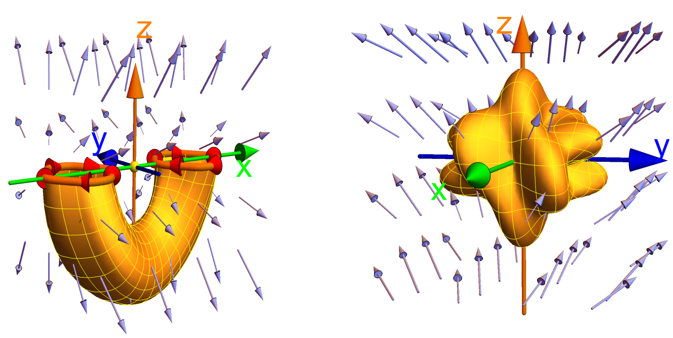


Figure 2. Stokes theorem and the Gauss theorem.

In the general frame work, the field $F = Pdx + Qdy + Rdz$ is a 1-form and the 2-form $dF = (P_x dx + P_y dy + P_z dz)dx + (Q_x dx + Q_y dy + Q_z dz)dy + (R_x dx + R_y dy + R_z dz)dz = (Q_x - P_y)dxdy + (R_y - Q_z)dydz + (P_z - R_x)dzdx$ is written as a column vector field $\text{curl}(F) = [R_y - Q_z, P_z - R_x, Q_x - P_y]^T$. To understand the flux integral, we need to see what a bilinear form like $dxdy$ does on the pair of vectors r_u, r_v. In the case $dxdy$ we have $dxdy(r_u, r_v) = x_u y_v - y_u x_v$ which is the third component of the cross product $r_u \times r_v$ with $r_u = [x_u, y_u, z_u]^T$. Integrating dF over S is the same as integrating the dot product of $\text{curl}(F) \cdot r_u \times r_v$. Stokes theorem implies that the flux of the curl of F only depends on the boundary of S. In particular, the flux of the curl through a closed surface is zero because the boundary is empty.

38.5. Gauss theorem: if the surface S bounds a solid E in space, is oriented outwards, and F is a vector field, then

Theorem: $\iiint_E \mathrm{div}(F)\, dV = \iint_S F \cdot dS$

Gauss theorem deals with a 2-form $F = Pdydz + Qdzdx + Rdxdy$, but because a 2-form has three components, we can write it as a **vector field** $F = [P, Q, R]^T$. We have computed $dF = (P_x dx + P_y dy + Q_z dz)dydz + (Q_x dx + Q_y dy + Q_z dz)dzdx + (R_x dx + R_y dy + R_z dz)dxdy$, where only the terms $P_x dxdydz + Q_y dydzdx + R_z dzdxdy = (P_x + Q_y + R_z)dxdydz$ survive which we associate again with the scalar function $\mathrm{div}(F) = P_x + Q_y + R_z$. The integral of a 3-form over a 3-solid is the usual triple integral. For a divergence free vector field F, the flux through a closed surface is zero. Divergence-free fields are also called **incompressible** or **source free**.

Remarks

38.6. We see why the 3 dimensional case looks confusing at first. We have three theorems which look very different. This type of confusion is common in science: we put things in the same bucket which actually are different: it is only in 3 dimensions that 1-forms and 2-forms can be identified. Actually, more is mixed up: not only are 1-forms and 2-forms identified, they are also written as vector fields which are T^1_0 tensor fields. From the tensor calculus point of view, we identify the three spaces $T^1_0(E) = E, T^0_1(E) = \Lambda^1(E) = E^*$ and $\Lambda^2(E) \subset T^0_2$. While we can still always identify vector fields with 1-forms, this identification in a general non-flat space will depend on the metric. In $\mathbb{R}^4$, the 2-forms have dimension 6 and can no more be written as a vector. One still does. The electro-magnetic F is a 2-form in $\mathbb{R}^4$ which we write as a pair of two time-dependent vector fields, the electric field E and the magnetic field B.

38.7. Geometries and fields are remarkably similar. On geometries, the **boundary operation** δ satisfies $\delta \circ \delta = 0$. On fields the **derivative operation** d satisfies $d \circ d = 0$. 'Geometries" as well as "fields" come with an **orientation**: $r_u \times r_v = -r_v \times r_u$, $dxdy = -dydx$. The operations d and δ look different because calculus deals with smooth things like curves or surfaces leading to generalized functions. In **quantum calculus** they are thickened up and d, δ defined without limit. Fields and geometries then become indistinguishable elements in a Hilbert space. The exterior derivative d has as an adjoint $\delta = d^*$ which is the boundary operator. It is a kind of quantum field theory as d generates while d^* destroys a "particle". $d^2 = \delta^2 = 0$ is a "Pauli exclusion".

38.8. We can spin this further: a m**-manifold** S is the image of a parametrization $r : G \subset \mathbb{R}^m \to \mathbb{R}^n$. The Jacobian dr is a **dual** m**-form**, the exterior product of the m vectors dr_{u_1} up to dr_{u_m} (think of m column vectors attached to $r(u) \in S$). If we take a map $s : S \subset \mathbb{R}^n \to \mathbb{R}^m$ and look at $F = ds$, we can think of it as a m-form F (think of m row vectors attached to each point x in $\mathbb{R}^n$). The map s defines $m \times n$ Jacobian $ds(x)$, while the **Jacobian** $dr(u)$ is the $n \times m$ matrix. Cauchy-Binet shows that the flux of $F = ds$ through $r(G) = S$ is the integral $\int_G F = \int_G \det(ds(r(u))dr(u))\ du$ $= \int_S \det(ds(x)dr(s(x)))$. If $s(r(u)) = u$, then this is a geometric functional. So: **geometries** G can come from maps from a space A to a space B, while **fields** F can come from maps from B to A. The **action integral** $\int_G F$ generalizes the Polyakov action $\int_G \det(dr^T dr) = \int_G |dr|^2$, a case where F and G are dual meaning $s(r(u)) = u$.

Prototype examples

Problem: Compute the line integral of $F(x,y,z) = [5x^4 + zy, 6y^5 + xz, 7z^6 + xy]$ along the path $r(t) = [\sin(5t), \sin(2t), t^2/\pi^2]$ from $t = 0$ to $t = 2\pi$.

Solution: The field is a gradient field df with $f = x^5 + y^6 + z^7 + xyz$. We have $A = r(0) = (0,0,0)$ and $B = r(2\pi) = (0,0,4)$ and $f(A) = 1$ and $f(B) = 4^7$. The **fundamental theorem of line integrals** gives $\int_C \nabla f \; dr = f(B) - f(A) = 4^7$.

Problem: Find the line integral of the vector field $F(x,y) = [x^4+\sin(x)+y+5xy, 4x+y^3]$ along the cardiod $r(t) = (1+\sin(t))[\cos(t), \sin(t)]$, where t runs from $t = 0$ to $t = 2\pi$.

Solution: We use Green's theorem. Since $\text{curl}(F) = 3 - 5x$, the line integral is the double integral $\iint_G 3 - 5x \; dxdy$. We integrate in polar coordinates and get $\int_0^{2\pi} \int_0^{1+\sin(t)} (3 - 5r\cos(t))r drdt$ which is $9\pi/2$. One can short cut by noticing that by symmetry $\iint_G (-5x) \; dxdy = 0$, so that the integral is 3 times the area $\int_0^{2\pi} (1+\sin(t))^2/2 \; dt = 3\pi/2$ of the cardioid.

Problem: Compute the line integral of $F(x,y,z) = [x^3 + xy, y, z]$ along the polygonal path C connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$.

Solution: The path C bounds a surface $S : r(u,v) = [u,v,0]$ parameterized on $G = \{(x,y)| \; x \in [0,2], y \in [0,1]\}$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(F)(x,y,z) = [0,0,-x]$ through S. The normal vector of S is $r_u \times r_v = [1,0,0] \times [0,1,0] = [0,0,1]$ so that $\int\int_S \text{curl}(F) \cdot dS = \int_0^2 \int_0^1 [0,0,-u] \cdot [0,0,1] \; dvdu = \int_0^2 \int_0^1 -u \; dvdu = -2$.

Problem: Compute the flux of the vector field $F(x,y,z) = [-x, y, z^2]$ through the boundary S of the rectangular box $G = [0,3] \times [-1,2] \times [1,2]$.

Solution: By the **Gauss theorem**, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \; dzdydx = (3-0)(2-(-1))(4-1) = 27$.