LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 1: Linear Spaces

Lecture

1.1. X is called a **linear space** over the real numbers $\mathbb{R}$ if there is an **addition** $+$ on X, a **zero element** in X and a **scalar multiplication** $x \to \lambda x$ with $1x = x$ in X. Additionally, we want that every x in X can be **negated**; this additive inverse element $-x$ satisfying $x + (-x) = 0$. The zero element 0 is required to satisfy $x + 0 = x$ for all x. With **addition**, we mean an operation which satisfies the **associativity** law $(x + y) + z = x + (y + z)$, the **commutativity** laws $x + y = y + x$, $\lambda x = x\lambda$ and the **distributivity** laws $\lambda(x + y) = \lambda x + \lambda y$, $\lambda\mu(x + y) = \lambda(\mu x + \mu y)$.

1.2. We are familiar with the real numbers $X = \mathbb{R}$. They form a linear space and we have learned to compute with these numbers early on like $7(3 + 5) = 56$. The rules of computation, like the associativity rule are not **results** which are proven but are considered **axioms** meaning that they are assumptions. There are simpler structures requiring less axioms: an example is the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3 \dots\}$, where we have no additive inverse. To have an additive inverse, we need to extend the natural numbers to $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

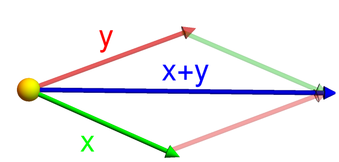
1.3. The set $M(n, m)$ is the space of all $n \times m$ matrices, arrays of numbers in which there are n rows and m columns. It is an example of a **linear space**: it contains a **zero element** in the form of the **0-matrix**. We can **add** $(A + B)_{ij} = A_{ij} + B_{ij}$, **subtract** $(A - B)_{ij} = A_{ij} - B_{ij}$ and **multiply with scalars** λA. An important class of matrices is the set $\mathbb{R}^n = M(n, 1)$ of **column vectors**. It is the n-dimensional **Euclidean space**. We especially like **the plane** $\mathbb{R}^2$ which we use for writing and $\mathbb{R}^3$, the space we live in.

Theorem: $X = M(n, m)$ is a linear space.

Proof. The associativity, commutativity and distributivity properties are inherited from the reals because each component A_{ij} is a real number. □

1.4. If we want to check whether a subset X of $M(n, m)$ is a linear space, the associativity, commutativity and distributivity properties are inherited from the ambient space. We now only need to check the following three properties:

i: $0 \in X$.
ii: $x + y \in X$ if $x, y \in X$.
iii: $\lambda x \in X$ if $x \in X$.



Theorem: If $X \subset M(n,m)$ satisfies $(i) - (iii)$, then X is linear.

Proof. The addition, scalar multiplication and zero element which satisfy the associative, commutative and distributive properties are given in $M(n,m)$. They hold therefore also in X. We only need to make sure therefore that addition, and scalar multiplication keeps us in X and also that 0 is in X and this is what the three conditions tell. □

1.5. Examples.
a) The set X of non-negative 3×4 matrices for example is not a linear space. Yes, it satisfies $i)$ and $ii)$ but property $iii)$ fails. For $\lambda = -1$, and a non-zero $A \in X$, the matrix λA is not in X.
b) The set X of all 3×4 matrices for which the sum of all matrix entries is zero is a linear space. Proof: i) check. The zero matrix entries add up to zero. ii) check. If $\sum_{i,j} A_{ij} = 0$ and $\sum_{i,j} B_{ij} = 0$, then $\sum_{i,j}(A+B)_{ij} = 0$. Finally, if A satisfying $\sum_{i,j} A_{ij} = 0$ and $\lambda \in \mathbb{R}$ are given then $\sum_{i,j}(\lambda A)_{ij} = \lambda \sum_{i,j} A_{ij} = 0$.

1.6. If there is also a multiplication $A \cdot B$ defined on a linear space X for which associativity, distributivity hold and a 1-element exists, one calls X an **algebra**. We do not require the multiplication to be commutative. Remember that the **matrix multiplication** $(AB)_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}$ was defined if $A \in M(n,p)$ and $B \in M(p,m)$. The result was a matrix in $M(n,m)$. If $m = n$, then the result is again in X.

Theorem: The space $M(n,n)$ is an algebra.

Proof. The identity matrix 1 satisfies $1 \cdot A = A$. We write 1 but note that 1 is an element in $M(n,n)$ and not an element in $\mathbb{R}$. We also write $\cdot$ for the multiplication if there can be some ambiguity like $2 \cdot 1$ is twice the identity matrix and not 21. The associativity property $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ can be established formally as both sides are $(ABC)_{k,l} = \sum_{i,j} A_{ki}B_{ij}C_{jk}$. The distributivity property $A \cdot (B+C) = A \cdot B + A \cdot C$ also can be checked with $(A \cdot (B+C))_{k,l} = \sum_j A_{kj}(B+C)_{jl} = \sum_j A_{kj}(B_{jl} + C_{jl}) = \sum_j A_{kj}B_{jl} + \sum_j A_{kj}C_{jl} = AB + AC$. □

1.7. We can compute with square matrices as with numbers. Here are three things which are different in the algebra $M(n,n)$.

The algebra $M(n,n)$ is not commutative if $n > 1$

1.8. For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, we have $AB = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

1.9.

There are infinitely many non-invertible elements if $n > 1$.

The 2×2 matrix with all entries being 1 is non-zero but not invertible.

1.10.

A^n can grow different than exponentially for $n > 1$.

In dimension 1, we have A^n either growing exponentially, decaying exponentially or staying bounded. For $n = 2$ already, we can have growth which is linear: for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

Examples

1.11. Let $A = \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Then $A+B = \begin{bmatrix} 5 & 5 \\ -1 & 2 \end{bmatrix}$ and $5A-3B = \begin{bmatrix} 9 & 17 \\ -5 & 2 \end{bmatrix}$ and $A^3 = \begin{bmatrix} -1 & 36 \\ -9 & -19 \end{bmatrix}$ and $7A^{-1} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}$.

1.12. Is the upper half plane $H = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ a linear space? It contains a zero element, is stable under addition and if v is there, also any multiple λv is there. Yes, almost. But it fails with scalar multiplication. $x \to \lambda x$ does not preserve H for negative λ. It is not a linear space.

1.13. Is the set X of 2×2 matrices for which $A_{11} = 0$ a linear space? Yes, it is. We check the three properties. The zero matrix is in the space. The sum of two matrices of this form is a matrix of this form. And if we multiply a matrix in X with a constant, then also λx is in X.

1.14. Is the set X of 2×2 matrices for which all entries are rational numbers a linear space? We can call this space $M_{\mathbb{Q}}(2,2)$. Check the properties. It is your turn.

1.15. Is the space $\{(x,y,z) \in \mathbb{R}^3 \mid xyz = 0\}$ a linear space?

Illustrations

1.16. Is the set X of all pictures with 800×600 pixels a linear space? We can add two pictures, multiply (make it brighter) and also have the zero picture, where red, green and blue entries are zero. While X is part of a linear space it is not a linear space itself. The pixel color range is an integer $[0, 255]$. Even if we allow real color values, the bounded range prevents X to become a linear space. But X is **part** of a linear space.

FIGURE 1. A is a flower from the garden of Emily Dickinson in Amherst. B is a portrait of Emily Dickinson herself. We then formed $0.3A + 0.7B$.

Homework

This homework is due on Tuesday, 2/6/2019.

Problem 1.1: Which of the following spaces are linear spaces?
a) the set of symmetric 2×2 matrices. ($A^T = A$).
b) the set of anti-symmetric 2×2 matrices. ($A^T = -A$).
c) the set of 2×2 diagonal matrices with zero trace.
d) the set of 2×2 matrices for which all entries are ≥ 0.
e) the set of 2×2 matrices with determinant 1.
f) the set of 2×2 matrices which are not invertible.
g) the set of 2×2 matrices which are in row reduced echelon form.
h) the set of 2×2 matrices with zero trace.

Problem 1.2: a) Take the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, then compute A^2, A^3 etc. Can you find a pattern for A^n?
b) Do the same for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Problem 1.3: a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ using row reduction.
b) Assume $A^{10} = 1$, can you find an expression for the inverse of A which only involves addition and multiplication?
c) Write down the inverse $(AB)^{-1}$ as a product of two inverses. Is it $A^{-1}B^{-1}$?

Problem 1.4: a) Solve the equation $AXB = 3X$ for X in the case $A = \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.
b) Find a 3×3 matrix A such that A, A^2 are not zero but A^3 is the zero matrix.

Problem 1.5: For any function $f(x)$ with a Taylor expansion $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and a matrix A we can also define $f(A) = \sum_{n=0}^{\infty} a_n A^n$.
a) Assume $f(x) = (1-x)^{-1}$, compute $f(\text{Diag}(1/2, 1/3))$ and the series.
b) Assume $f(x) = e^x$. Compute $f(\begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix})$ using the Taylor expansion.
c) Find a matrix $A \in M(2,2)$ which satisfies $\sin(A) = 0$ but which is not of the form $A = 0$ or $A = \pi = \pi I_2$.