Unit 2: Linear transformations

Lecture

2.1. A matrix $A \in M(n,m)$ defines a **linear map** $T(x) = Ax$ from $\mathbb{R}^m$ to $\mathbb{R}^n$. For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, then $Av = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The matrix A defines a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$.

2.2. To verify that a map T is linear, we have to check three things:

$T(0) = 0$	compatibility with zero
$T(x+y) = T(x) + T(y)$	compatibility with addition
$T(\lambda x) = \lambda T(x)$	compatibility with scaling

2.3. The vector $e_k = [0, \dots, 0, 1, 0, \dots, 0]^T \in \mathbb{R}^n$ is called the k'th **basic basis vector**. We can also see the vector e_k is the k'th column of the identity matrix 1 in $M(n,n)$.

2.4. An important link between geometry and algebra is the following simple fact.

Theorem: The k'th column vector of A is the image of e_k.

Proof. Just compute the image of e_k which is Ae_k. Formally, we can just write $(Ae_k)_i = \sum_j A_{ij}(e_k)_j = A_{ik}$, where we have used that $(e_k)_j = 1$ if $k = j$ and 0 else. □

2.5. This theorem is pivotal as it implies that if a transformation T satisfies the three properties above, then there is a matrix A which has the property that $T(v) = Av$. The proof is constructive in that the matrix A is explicitly given by its column vectors. The linearity then assures that if $v = [a_1, \dots, a_m]^T = a_1 e_1 + \cdots + a_m e_m$, then $Tv = a_1 T(e_1) + \cdots + a_m T(e_m)$ is the matrix multiplication Av. Since the $T(e_k)$ are columns, one calls this the **column picture**.

2.6. To see for example what the effect of $T(x) = Ax$ with $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is, we look at the column vectors. They have the same length and are perpendicular to each other. The transformation is a rotation dilation about the z-axes as rotation.

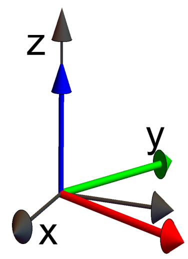

FIGURE 1. A rotation dilation in space. We link the transformation with the matrix by looking at the image of the basis vectors.

2.7. Here are the 4 most important types of linear transformations in the plane $\mathbb{R}^2$. Shear means "horizontal shear".

Rotation	Reflection	Projection	Shear
$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$	$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$	$\begin{bmatrix} \cos^2(\alpha) & \cos(\alpha)\sin(\alpha) \\ \sin(\alpha)\cos(\alpha) & \sin^2(\alpha) \end{bmatrix}$	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$

2.8. When combined with a dilation, the structure of the matrices becomes simpler: allowing dilations is simpler.

Rotation Dilation	Reflection Dilation	Projection Dilation	Shear Dilation
$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$	$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$	$\begin{bmatrix} a^2 & ab \\ ba & b^2 \end{bmatrix}$	$\begin{bmatrix} a & c \\ 0 & a \end{bmatrix}$

FIGURE 2. What kinds of transformations are these? In each of the two cases, we see the original square on the left and the transformed square to the right.

These four examples allow for building more complicated linear transformations.

2.9. A linear transformation $T : X \to X$ is called **invertible** if there exists another transformation $S : X \to X$ such that $TS(x) = x$ for all x.

Theorem: If T is linear and invertible, then T^{-1} is linear and invertible.

Proof. To invert $T(x) = Ax$, we have to be able to solve $Ax = b$ uniquely for every b. Let B be the matrix in which the k'th column b_k is the solution of $Ax = e_k$. Then $S(e_k) = b_k$ so that $ABx = x$. □

Examples

2.10. Is the map $T(x) = x + 1$ a linear transformation from $\mathbb{R}$ to $\mathbb{R}$? By definition, we would have to find a matrix A such that $T(x) = Ax$. But a 1×1 matrix A is just a real number. We would have to find a real number a such that $x + 1 = ax$ for all x. For $x = 0$ this gives $1 = 0$ which is not possible.

Illustrations

2.11. The tesseract is a 4 dimensional cube.

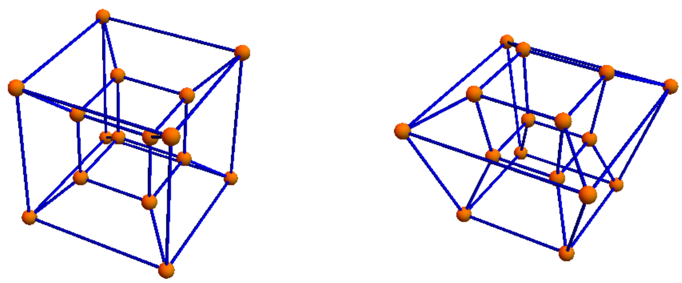

Figure 3. We see here the effect of a rotation in four dimensional space. The tesseract is first rotation by $\pi/3$, then again by $\pi/3$. The picture shows a projection into three dimensions.

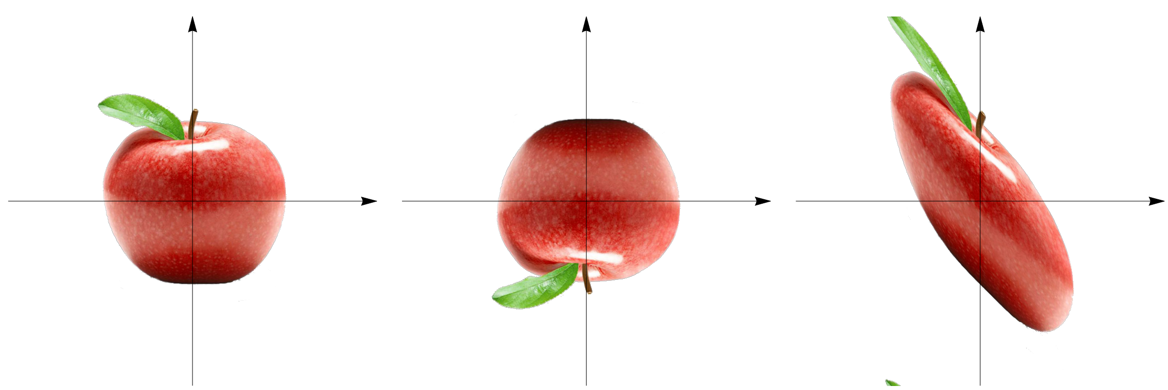

Figure 4. Can you find which of the matrices A-E implements the transformation from the left apple to the middle one. Which one implements the transformation from the left apple to the right one?

2.12. A=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ B=$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ C=$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ D=$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ E=$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Homework

This homework is due on Tuesday.

Problem 2.1: Which of the following transformations are linear?
a) $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by $T([x, y, z, w]^T) = [y, z]^T$.
b) $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by $T([x, y, z, w]^T) = [y + 1, w + x]^T$.
c) $T : M(3, 3) \to \mathbb{R}$ the map $T(A) = A_{22}$
d) $T : M(3, 4) \to M(4, 3)$ given by $T(A) = A^T$.
e) $T : M(2, 2) \to \mathbb{R}$ given by $T(A) = \det(A)$.
f) $T : M(2, 2) \to M(2, 2)$ given by $T(A) = A^2$.
g) $T : \mathbb{R} \to M(2, 2)$ by $T(\alpha) = R(\alpha)$, the rotation matrix by α.

Problem 2.2: a) What is the composition of two reflections at lines.
b) What is the composition of two rotation dilations.
c) What is the composition of two horizontal shears.
d) Verify that for a projection P, the identity $P^2 = P$ holds.
e) Verify that for a reflection T, the inverse is equal to T.
f) Find a composition of two projections A, B such that $AB = 0$.
g) What is the composition of a reflection and a rotation?

Problem 2.3: a) Find the matrix in $M(3, 3)$ which reflects about the x-axes.
b) Find the matrix in $M(3, 3)$ which maps $i = [1, 0, 0]$ to $j = [0, 1, 0]$ and $j = [0, 1, 0]$ to $k = [0, 0, 1]$ and maps $k = [0, 0, 1]$ to $i = [1, 0, 0]$.
c) Let A be the matrix in $M(2, 2)$ which reflects about the x-axes and B be the matrix which reflects about the y axes. Check that $AB = BA$.
d) Is it true in general that for two reflections A, B, the identity $AB = BA$ holds?
e) Find a 2×2 matrix which is neither a reflection, nor a rotation, nor a shear nor a projection.

Problem 2.4: a) Write down the matrix A of a rotation about the z-axes by an angle θ.
b) Write down the matrix B of a rotation about the y axes by an angle ϕ.
c) Compute AB. What is the vector $AB([1, 0, 0]^T)$? Do you recognize the coordinates of this vector?

Problem 2.5: Find the 4×4 matrix A which implements the multiplication of a quaternion z by $a + ib + jc + kd$.