

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 4: Basis and dimension

LECTURE

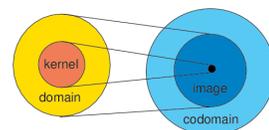
4.1. Let X be a linear space. A collection $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ of vectors in X **spans** X if every x in X can be written as a **linear combination** $x = a_1v_1 + \dots + a_nv_n$. The set \mathcal{B} is called **linearly independent** if $a_1v_1 + \dots + a_nv_n = 0$ implies that all a_i are zero. The set \mathcal{B} is a **basis** if it both spans X and if it is linearly independent.

4.2. For $X = \mathbb{R}^2$, three vectors $v_1 = [1, 1]^T$, $v_2 = [2, 3]^T$ and $v_3 = [0, 1]^T$ span the plane but they are not linearly independent because $v_1 - v_2 + v_3 = 0$. Indeed, we only need two vectors to span the entire plane. Already $\mathcal{B} = \{v_1, v_2\}$ spans the plane. They are also linearly independent because $a_1v_1 + a_2v_2 = 0$ would mean the two vectors are parallel. So, a collection of vectors in \mathbb{R}^2 spans if and only if it is not contained in a line. Similarly, a collection of vectors in $X = \mathbb{R}^3$ spans X if and only if the vectors are not contained in a plane.

4.3. A natural basis of \mathbb{R}^2 is given by the vectors $[1, 0]^T$ and $[0, 1]^T$. The collection $\mathbb{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is called the **standard basis** of the plane. The standard basis in \mathbb{R}^3 is $\mathbb{B} = \{i = e_1, j = e_2, k = e_3\}$. The standard basis in the quaternion space is $\mathbb{H} = \mathbb{R}^4$ is $e_1 = 1, e_2 = i, e_3 = j, e_4 = k$.

4.4. The **kernel** of a $n \times m$ matrix A is the set $\ker(A) = \{x \in \mathbb{R}^m \mid Ax = 0\}$. The **image** of A is the set $\text{im}(A) = \{Ax \mid x \in \mathbb{R}^m\} \subset \mathbb{R}^n$. These are important constructs because every linear space can be written either as a kernel or as an image of some matrix. Remember that we wrote a two-dimensional plane containing the origin as $ax + by + cz = 0$. This is the kernel of a matrix $A = [a, b, c]$. We also learned how to write the plane as a linear combination of two vectors v, w . The matrix A which contains these two vectors as column vectors has this plane as the image.

Lemma: Both the kernel and image of a matrix are linear spaces.



Proof. a) 0 is the kernel because $A0 = 0$. If x and y are in the kernel, then $Ax = 0$ and $Ay = 0$ implying $A(x + y) = 0$. So, $x + y$ is in the kernel. Also, if $Ax = 0$, then $A(\lambda x) = \lambda(Ax) = 0$ so that λx is in the kernel.

b) Zero is in the image because $A0 = 0$. If $x = Au$ and $y = Av$, then $x + y = Au + Av = A(u + v)$ so that $x + y$ is in the image. If $x = Au$, then $\lambda x = A(\lambda u)$ so that also λx is in the image of A . \square

4.5. How do we check whether a set of vectors is a basis? The key is the **transformation matrix** S which contains the elements of \mathcal{B} as columns. Given a matrix S , the **image** of S is the set of all vectors Sx , where x is in X . The **kernel** of S is the solution set of $Ax = 0$.

Lemma: \mathcal{B} is linearly independent if and only if S has a trivial kernel.

Lemma: \mathcal{B} spans X if and only if the image of S contains X .

4.6. We can find the kernel and image of a matrix by row reduction: the kernel is parametrized by free variables. The image is spanned by the columns of A . Here is something one at first does not appreciate. In principle, it would be possible that there is one basis of X with m vectors and a basis with n vectors. In principle, it would be possible that we can implement X as a space of vectors with m components and implement X differently as a space of vectors with n components. But this is not the case:

Theorem: Every basis of X has the same number of elements.

Proof. (i) We first show that if $\mathcal{A} = \{v_1, \dots, v_q\}$ span and $\mathcal{B} = \{w_1, \dots, w_p\}$ are linearly independent, then $q \geq p$.

Assume $q < p$. Because the v_j span, each vector w_i can be written as $w_i = \sum_{j=1}^q a_{ij}v_j$.

Now row reduce the augmented $(p \times (q + n))$ -matrix $\left[\begin{array}{ccc|c} a_{11} & \dots & a_{1q} & w_1^T \\ \dots & \dots & \dots & \dots \\ a_{p1} & \dots & a_{pq} & w_p^T \end{array} \right]$, where

v_i^T is the vector v_i written as a row vector. Each row of A of this matrix contains some nonzero entry. We end up with a matrix which contains a last row $\left| 0 \quad \dots \quad 0 \quad \left| b_1w_1^T + \dots + b_qw_q^T \right. \right|$ showing that $b_1w_1^T + \dots + b_qw_q^T = 0$. Not all b_j are zero because we had to eliminate some nonzero entries in the last row of A . This nontrivial relation of w_i contradicts linear independence. The assumption $q < p$ can not be true.

(ii) Because \mathcal{A} spans X and \mathcal{B} is linearly independent, we know that $q \leq p$. Because \mathcal{B} spans X and \mathcal{A} is linearly independent also $p \leq q$ holds. Together, $p \leq q$ and $q \leq p$ implies $p = q$. \square

4.7. It follows that every linear space X which is spanned by finitely many vectors has an integer attached to it. The **dimension** of a linear space is defined as the number of basis elements for a basis.

4.8. The dimension of the image of a matrix A is called the **rank** of A . The dimension of the kernel of a matrix A is called the **nullity** of A . The nullity of A is the number of columns without leading 1 in $\text{rref}(A)$; the rank is the number of columns with leading 1. The following theorem is called the **rank-nullity theorem** or **fundamental theorem of linear algebra**.

Theorem: $\text{rank}(A) + \text{nullity}(A) = n$

Proof. There are n columns. Either a column has no leading 1 after row reduction, leading to a free variable and so to a basis vector in the kernel, or then the column will lead to a leading one, meaning that the original column is in the image of A . \square

4.9. Can you write the basis for the kernel of the partitioned 4×4 matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in terms of a basis $\{v_1, v_2\}$ of the kernel of A and the basis $\{w_1, w_2\}$ of the kernel of the 2×2 matrix B . Answer: $\mathcal{B} = \left\{ \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \begin{bmatrix} v_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ w_1 \end{bmatrix}, \begin{bmatrix} 0 \\ w_2 \end{bmatrix} \right\}$.

EXAMPLES

4.10. The kernel and image of the projection matrix A which projects onto the plane $\Sigma : x + y + z = 0$ can be seen geometrically: the kernel of the transformation consists of all vectors orthogonal to Σ it has a basis $\mathbb{B} = \{[1, 1, 1]^T\}$, the image is the plane Σ . It has a basis $\mathbb{B} = \{[1, -1, 0]^T, [1, 0, -1]^T\}$.

4.11. A basis of the kernel of $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is $\{e_1, e_2\}$. A basis for the image is $\{e_1\}$. This is an example where the image is part of the kernel.

4.12. Problem: Find a basis for the image and a basis for the kernel of $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 9 & 12 & 16 \end{bmatrix}$.

Solution: We have $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There are two leading 1's. The rank of the matrix is 2, the nullity is 3. The basis of the image is given by the first and last column of the original matrix A :

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \\ 16 \end{bmatrix} \right\}.$$

To get a basis for the kernel, introduce free variables for the columns which have no leading 1 and write down the equations $x_1 + 2u - 3v - 4w = 0, x_2 = u, x_3 = v, x_4 = w, x_5 = 0$ then write down the solution in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = u \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

A basis of the kernel is

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

HOMEWORK

This homework is due on Tuesday, 2/13/2019.

Problem 4.1: Check whether the given set of vectors is linearly independent. Use the S -matrix to find out.

a) $\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$. b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

c) $\left\{ \begin{bmatrix} 3 \\ 16 \end{bmatrix}, \begin{bmatrix} 4 \\ 18 \end{bmatrix}, \begin{bmatrix} 5 \\ 19 \end{bmatrix} \right\}$. d) $\{[1], [2]\}$.

Problem 4.2: Find a basis for the image as well as a basis for the kernel of the following matrices

a) $\begin{bmatrix} 7 & 0 & 7 \\ 2 & 3 & 8 \\ 9 & 0 & 9 \\ 5 & 6 & 17 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. d) $[1 \ 2 \ 3 \ 4]$.

Problem 4.3: a) Find a basis for the kernel and a basis for the image of the projection dilation matrix

$$A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}.$$

b) Under which conditions do reflection dilations or rotation dilations or shear dilations all have a trivial kernel?

Problem 4.4: Look up unit 5 of Math 22a. We have seen there the theorem that the image of A^T is perpendicular to the kernel of A . Understand and write down the proof of this important result in your own words.

Problem 4.5: Write down statements which are equivalent to the fact that A is an invertible matrix:

- A statement involving the rank of A .
- A statement involving the nullity of A .
- A statement involving the linear independence of the columns of A .
- A statement involving the term basis and column vectors.