Unit 5: Change of Coordinates

Lecture

5.1. Given a basis $\mathcal{B}$ in a linear space X, we can write an element v in X in a unique way as a sum of basis elements. For example, if $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a vector in $X = \mathbb{R}^2$ and $\mathcal{B} = \{v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}\}$, then $v = 2v_1 + v_2$. We say that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}}$ are the $\mathcal{B}$ **coordinates** of v. The **standard coordinates** are $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ are assumed if no other basis is specified. This means $v = 3e_1 + 4e_2$.

5.2. If $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ is a basis of $\mathbb{R}^n$, then the matrix S which contains the vectors v_k as column vectors is called the **coordinate change matrix**.

Theorem: If S is the matrix of $\mathcal{B}$, then $S^{-1}v$ are the $\mathcal{B}$ coordinates of v.

5.3. In the above example, $S = \begin{bmatrix} 1 & 1 \\ -1 & 6 \end{bmatrix}$ has the inverse $S^{-1} = \begin{bmatrix} 6 & -1 \\ 1 & 1 \end{bmatrix}/7$. We compute $S^{-1}[3,4]^T = [2,1]^T$.

Proof. If $[v]_{\mathcal{B}} = [a_1, \dots, a_n]$ are the new coordinates of v, this means $v = a_1v_1 + \cdots + a_nv_n$. But that means $v = S[v]_{\mathcal{B}}$. Since $\mathcal{B}$ is a basis, S is invertible and $[v]_{\mathcal{B}} = S^{-1}v$. $\square$

Theorem: If $T(x) = Ax$ is a linear map and S is the matrix from a basis change, then $B = S^{-1}AS$ is the matrix of T in the new basis $\mathcal{B}$.

Proof. Let $y = Ax$. The statement $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$ can be written using the last theorem as $S^{-1}y = BS^{-1}x$ so that $y = SBS^{-1}x$. Combining with $y = Ax$, this gives $B = S^{-1}AS$. $\square$

5.4. If two matrices A, B satisfy $B = S^{-1}AS$ for some invertible S, they are called **similar**. The matrices A and B both implement the transformation T, but they do it from a different perspective. It makes sense to adapt the basis to the situation. For example, here on earth, at a specific location, we use a coordinate system, where v_1

points east, where v_2 points north and where v_3 points straight up. The natural basis here in Boston is different than the basis in Zürich. [1]

Figure 1. A good coordinate system is adapted to the situation. When talking about points on the globe, we can use a global coordinate system with e_3 in the earth axes. When working on earth say near Boston, we need another basis.

5.5. Using a suitable basis is one of the main reasons why linear algebra is so powerful. This idea will be a major one throughout the course. We will use "eigen basis" to diagonalize a matrix, we will use good coordinates to solve ordinary and partial differential equations.

5.6. For us, the change of coordinates now is a way to figure out the matrix of a transformation

> To find the matrix A of a reflection, projection or rotation matrix, find a good basis for the situation, then look what happens to the new basis vectors. This gives B. Now write down the matrix S and get $A = SBS^{-1}$.

Examples

5.7. Problem. Find the matrix A which implements the reflection T at the plane $X = \{x + y + 2z = 0\}$. **Solution.** We take a basis adapted to the situation. Take $v_3 = [1, 1, 2]^T$ which is perpendicular to the plane, then choose $v_1 = [1, -1, 0]^T, v_2 = [2, 0, -1]^T$ which are in the plane. Now, since $T(v_1) = v_1, T(v_2) = v_2$ and $T(v_3) = -v_3$, the transformation is described in that basis with the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The

[1] Figure (1) was rendered using creative commons Povray code by Gilles Tran authored 2000-2004.

basis change transformation S is $S = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$. We can now get $A = SBS^{-1} = \begin{bmatrix} -1 & -2 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

5.8. Problem. Find the matrix which rotates about the vector $[3,4,0]^T$ by 90 degrees counter clockwise when looking from the tip $(3,4,0)$ of the vector to the origin $(0,0,0)$. **Solution.** We build a basis adapted to the situation. Of course, we use $v_1 = [3,4,0]^T$. We need now two other vectors which are perpendicular to each other. The vectors $v_2 = [-4,3,0]^T$ and $v_3 = [0,0,5]$ present themselves. It is good to have the two vectors which are moving to have the same length because then the matrix B is particularly simple: since $v_2 \to -v_3, v_3 \to v_2, v_1 \to v_1$, we have $B = \begin{bmatrix} 1 & 0 & 0 \\ x0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. With $S = \begin{bmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, we get $A = SBS^{-1}$. This is now just matrix multiplication and can be computed $\begin{bmatrix} 9 & 12 & -100 \\ 12 & 16 & 75 \\ 4 & -3 & 0 \end{bmatrix}/25$. It would have been quite hard to find the column vectors of this matrix by figuring out where each of the standard basis vectors e_k goes. Still, we have used that basic principle when figuring out what B is.

5.9. Find the matrix of the projection on the line perpendicular to the hyperplane $x+y+z+w=0$ in $\mathbb{R}^4$. Solution: there is a nice basis adapted to that situation. It gives

$$\mathcal{B} = \{v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} v_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\}, S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

S is invertible. In this case $S^{-1} = 4S$. Now, in the new basis, the transformation matrix is very simple. As v_1 goes to v_1 and v_2 and v_3 and v_4 all go to zero, we have

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $A = SBS^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}/4$. In this case, we might also have been able to write down the matrix without going to a new coordinate system as the image of the first basis vector is the vector projection of $[1,0,0,0]$ onto $[1,1,1,1]$.

Homework

This homework is due on Tuesday, 2/13/2019.

Problem 5.1: What are the $\mathcal{B}$-coordinates of $\vec{v}$ in the basis $\mathcal{B}$.

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \mathcal{B} = \{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \} ?$$

Problem 5.2: What is the matrix B for the transformation $A = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ in the basis $\mathcal{B} = \{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$:

Problem 5.3: Chose a suitable basis to solve:
a) What matrix A implements the reflection at the plane $3x+3y+6z = 0$?
b) What matrix A implements the reflection at the line spanned by $[2, 2, 4]^T$?

Problem 5.4: Find the matrix A corresponding to the orthogonal projection onto the plane spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Problem 5.5: "Graphene" are hexagonal planar structures. We can work with them when using a good adapted basis. Assume the first is $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. a) Find w so that $\mathcal{B} = \{v, w\}$ is the basis as seen in the picture.
b) What are the standard coordinates of $\begin{bmatrix} 3 \\ -1 \end{bmatrix}_{\mathcal{B}}$?
c) Is $\begin{bmatrix} 23 \\ 72 \end{bmatrix}_{\mathcal{B}}$ a vertex of a hexagon or the center of one?

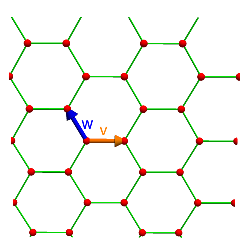

Figure 2. Graphene are single layer hexagonal lattice carbon structures.

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