Unit 6: Prototypes

Seminar

6.1. One of the powerful mind-hacks of **Richard Feynman** was his ability to **use examples** to make a point. He was known for acute sharpness to spot errors in arguments and once explained what was the secret behind that: rather than following the general abstract presentation, he would make up a basic example and project the general case to that. The principle is simple: if the example fails, then the general theory fails. While examples do not cover an entire theory, they serve as **prototypes** which capture the theory. We want to explore this today.

Figure 1. Richard Feynman playing bongo drums (1962).

1

[1]We colorized the picture and enlarged part to get the right picture. The book "Feynman Tips on Physics" uses the left picture as a cover and states that the photographer is unknown. The widespread use, both commercially as well as in picture collections, appears to put the photo into the public domain.

6.2. We explore today the power of examples. Examples not only provide **counter examples** to statements we believe to be true, examples also are **prototypes** for general results. In many cases, one can reduce the general case to a prototype. Feynman was good at listening to a general statement, then running it through a well chosen example. This immediately allowed him to spot errors simply because the structure failed in the example. This principle is in particular powerful in linear algebra.

6.3. Let's look at the example of a reflection at a line. We have derived in unit 2, why the matrix of a reflection at a line containing the vector $[\cos(\alpha), \sin(\alpha)]$ was the matrix $A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$. A special case is if we deal with the x-axis. In that case, the matrix is $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

6.4. Let's look at the statement

Theorem: A reflection at a line preserves length and angles.

Problem A: Look at a simple special case to verify that it works if the basis is chosen in an appropriate way.

6.5. Checking the result in a special coordinate system is simpler than verifying the theorem in full generality by comparing the lengths of

$$v = \begin{bmatrix} x \\ y \end{bmatrix}, Av = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and then checking that the dot product between $[x, y]^T$ and $[u, v]^T$ is the same as the dot product between $A[x, y]^T$ and $A[u, v]^T$. For the later, there is an elegant reduction to length:

Problem B: Verify that if one can measure length $|x|$ in a linear space, then one can also get the dot product. Hint: compute $|x - y|^2, |x + y|^2$ and $|x|^2$ and $|y|^2$ and see how to get $x \cdot y$.

6.6. Why does the fact that a reflection preserves the dot product imply that angles are preserved?

Problem C: How again can one get the angle between two vectors from the dot product?

6.7. Here is a tougher question: is it possible that for a 2×2 matrix A and a vector x, the length of $A^n x$ grows quadratically?

Problem D: Explore the quadratic growth question with the eyes of Feynman. We don't have the tools yet to answer this definitely, but we want you to try finding a 2×2 matrix, where this happens. As you don't succeed finding a 2×2 matrix, can you find a 3×3 matrix where quadratic growth happens?

6.8. In any subject, whether it is science, art, law or humanities, it is good to have a collection of prototypes available. One should have examples which are **typical** and also have examples which are **unusual** and **special**. The latter serve as counter examples or examples of surprising things. Here are some examples in linear algebra:

- $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is an example of a matrix which has the property that $A^2 = -1$. This can not be realized in the real numbers.
- $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is an example of a matrix which has the property that $A^2 = 0$. This can not be realized in the real numbers without being zero. It is also an example, for which the image of A is the kernel of A.

6.9. Now we can ask ourselves, what is the collection of all the 2×2 matrices which have the property that $A^2 = -1$? This is a tougher question but we can easily give more examples by changing basis. If $B = S^{-1}AS$, then $B^2 = S^{-1}ASS^{-1}AS = S^{-1}A^2S = S^{-1}(-1)S = -S^{-1}S = -1$. We can also just by brute force, look for $A^2 = -1$ and see that $d = -a$ and $cb = -1 - a^2$. in the second case $A^2 = 0$, we get $d = -a$ and $cb^2 = -a^2$.

6.10. Let $r(t) = [x(t), y(t)]$ denote a closed curve. To play **billiards** inside this table, we reflect a ball with the **reflection rule** telling that the incoming angle is equal to the outgoing angle.

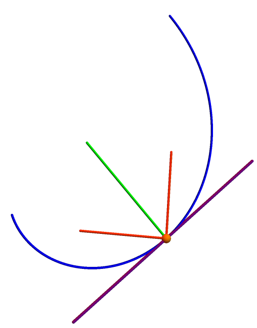

FIGURE 2. How do we compute the reflection at the table boundary?

Assume we have $r(t) = [\sin(t) + t^2, t^4 + t]$ and want to compute the matrix A which reflects in coming ball with velocity v at the point $r(0)$. The outgoing vector should be Av.

Problem E: What is a good basis for this problem? How does the reflection matrix B look like in this coordinate system?

Problem F: If you have time, find the matrix A.

HOMEWORK

Exercises A)-D) are done in the seminar. This homework is due on Tuesday:

Problem 6.1 a) Is it true that if A is an invertible 2×2 matrix with rational entries, then A^{-1} is a 2×2 matrix with rational entries? If yes, give a proof.
b) Is it true that if A is an invertible 2×2 matrix, where all entries are 1 or 0, then A^{-1} is a 2×2 matrix, where all entries are 0 or 1.
c) Is it true that if A is an invertible 2×2 matrix where all entries are 1 or 0, then A^{-1} is a 2×2 matrix, where all entries are 0 or 1 or -1.

Problem 6.2 a) Is it true that if A is an invertible $n \times n$ matrix with rational entries, then A^{-1} is a $n \times n$ matrix with rational entries. b) Is it true that if A is an invertible $n \times n$ matrix where all entries are 1 or 0, then A^{-1} is a $n \times n$ matrix where all entries are 0 or 1 or -1.

Problem 6.3 A matrix is called **unimodular**, if its determinant is 1.
a) Is it true that if A is a unimodular 2×2 matrix with integer entries, then its inverse is a 2×2 matrix with integer entries?
b) Is it true that if A is a unimodular 3×3 matrix with integer entries, then its inverse is a 3×3 matrix with integer entries?

Problem 6.4 Let's look at a plane $ax + by + cz = 0$, where a, b, c are integers in which not all are zero.
a) Is it true that the matrix of reflection at the plane has integer entries?
b) Is it true that the matrix of reflection at the plane has rational entries?
c) Is it true that the matrix of orthogonal projection onto the plane has integer entries?
d) Is it true that the matrix of orthogonal projection onto the plane has rational entries?

Problem 6.5 Experiment with a computer algebra system. Take a random matrix and take its inverse and then form its exponential. Make some pictures like $MatrixPlot[MatrixExp[Table[Random[], \{100\}, \{100\}]]]$ Explore using experiments, whether you can find A where $\exp(A)$ or A^{-1} look random.