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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 13: First Hourly Practice

Welcome to the first hourly. It will take place on February 26, 2019 at 9:00 AM sharp in Hall D. You can already fill out your name in the box above.

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. We want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it.
- You have 75 minutes for this hourly.

PROBLEMS

Problem 13P.1 (10 points):

We prove $\text{im}(A) = \text{im}(AA^T)$ in two steps. Do them both.

- a) Prove that $\text{im}(A)$ contains $\text{im}(AA^T)$.
- b) Prove that $\text{im}(AA^T)$ contains $\text{im}(A)$.

Solution:

a) The image of AA^T is contained in the image of A because we can write $v = AA^T x$ as $v = Ay$ with $y = A^T x$.

b) On the other hand, if v is in the image of A , then $v = Ax$. If $x = y + z$, where y in the kernel of A and z orthogonal to the kernel of A , then $Ax = Az$. Because z is orthogonal to the kernel of A , it is in the image of A^T . Therefore, $z = A^T u$ and $v = Az = AA^T u$ is in the image of AA^T .

Problem 13P.2 (10 points):

Decide in each case whether the set X is a linear space. If it is, prove it, if it is not, then give a reason why it is not.

- a) The space of 4×4 matrices with zero trace.
- b) The space of 4×4 quaternion matrices.
- c) The space of 2×2 matrices λQ , where Q is an orthogonal matrix and λ is real.
- d) The space of 3×3 matrices λQ , where Q is an orthogonal matrix and λ is real.
- e) The space of 5×5 matrices with entries 0 or 1.
- f) The image $\text{im}(1)$ of the identity 3×3 matrix 1 .
- g) The kernel of the matrix $A = [1, 2, 3, 4]$.
- h) The set of all vectors in \mathbb{R}^3 for which $x^2 + y^2 + z^2 = 0$.
- i) The set of all vectors in \mathbb{R}^3 for which $xyz = 0$.
- j) The set of 3×3 matrices with orthogonal columns.

Solution:

a) Yes. Proof: 0 is in X . Since $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ also the sum is in X . Since $\text{tr}(\lambda A) = \lambda \text{tr}(A)$, also a scalar multiple of A is in X .

b) Yes. you have even found a basis for it. $1a + ib + jc + kd$, where $1, i, j, k$ are specific 4×4 matrices. The zero quaternion is the zero matrix. Quaternions can be added and scaled.

c) No, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ The sum is no more in the space. Note that if we would have taken for Q a rotation matrix (determinant 1), then the answer would have been yes. It would have been the set of rotation-dilation matrices, which form a linear space.

d) No, Take a similar counter example as in c like $A = \text{Diag}(1, 1, 1)$ and $B = \text{Diag}(1, -1, -1)$.

e) No, Take $A = 1$ which is in the X but $2A = 2$ is not in X . (As a side remark, it would be a linear space over the Boolean field $Z_2 = \{0, 1\}$ but for us, we take real vector spaces unless stated otherwise.)

f) yes, The image of a matrix is always a linear space.

g) Yes, The kernel of a matrix is always a linear space.

h) Yes, This is the trivial space $X = \{0\}$. Yes, it is a linear space.

i) No, we can not add in this space like $[1, 0, 0]^T + [0, 1, 1]^T = [1, 1, 1]^T$ is no more in the

space. j) No, take $A = 1$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The sum $A + B$ does not have orthogonal columns any more.

Problem 13P.3 (10 points):

- a) (2 points) Is the transformation $T(x, y) = (x^2, y^2)$ from \mathbb{R}^2 to \mathbb{R}^2 linear?
 b) (2 points) Is the map $T(A) = \text{tr}(A)1$ as a map from $M(3, 3)$ to $M(3, 3)$ linear? Here 1 is the identity matrix.

c) (4 points) Match the transformation type:

A: **rotation dilation**, B: **reflection dilation**, C: **shear dilation**, D: **projection dilation**

Fill A-D				
Matrix	$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

d) (4 points) *SO, SU* or *NO* (no *SO* nor *SU*)? Yo, you have to decide so!

SO,SU,NO				
Matrix	$\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} i & i \\ i & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Solution:

- a) No, $T(2x, 2y) \neq 2T(x, y)$.
 b) Yes. $T(0) = 0$, $T(x + y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$.
 c) Projection dilation, rotation dilation, reflection dilation, shear dilation
 d) NO. This is i times a rotation dilation matrix. The determinant is not 1. NO. The determinant is zero NO. The determinant is zero SU. The determinant is 1.

Problem 13P.4 (10 points, each sub problem is 2 points):

- a) Which physicist promoted the use of examples to understand a theory?
 b) We describe a general linear map T from $M(2, 2)$ to $M(2, 2)$. In a suitable basis, is this map given by a 2×2 matrix or a 4×4 matrix?
 c) Is it true that $f(x) = e^{3x}$ satisfies $f(x) = O(e^x)$?
 d) Give an example of a 4×4 matrix A for which $\ker(A) = \text{im}(A)$.
 e) If $B = S^{-1}AS$, what can you say about the determinants of A and B ?

Solution:

- a) Feynman. b) 4×4 matrix. The space $M(2, 2)$ is equivalent to R^4 .
 c) No. It is true that $e^{O(3x)} = e^{O(x)}$.
 d) Take a partitioned matrix containing two 2×2 magic matrix A in the diagonal implementing $T(x, y) = (0, x)$.
 e) The determinants are the same.

Problem 13P.5 (10 points):

In the **checkers matrix**, the entry 1 means that the checkers initial condition has a checker piece there and 0 means that that field is empty:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

- a) (6 points) Find a basis for the kernel of A .
 b) (4 points) Find a basis for the image of A .

Solution:

a) The kernel is 6 dimensional. To find it, row reduce the matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Introduce free variables a, b, c, d, e, f and write down the equations $x = -a - c, y = -b - d, z = a, y = b, v = c, w = d, p = e, q = f$ to get

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \\ p \\ q \end{bmatrix} = a \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so that

$$\mathcal{B}_{ker}(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

b) the image 2 dimensional. It is spanned by the first two vectors in the checkers matrix, because that's where the leading 1 were in $rref(A)$.

$$\mathcal{B}_{im}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 13P.6 (10 points):

a) (5 points) The projection-dilation matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ in the basis

$$\mathcal{B} = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is given by a matrix B . Find this 3×3 matrix B .

b) (5 points) A linear transformation T satisfies

$$T(v_1) = v_2, T(v_2) = v_3, T(v_3) = v_1$$

where v_1, v_2, v_3 are given in a). Find the matrix R implementing this transformation in the standard basis.

Solution:

a)

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B = S^{-1}AS = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

b) $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, R = SBS^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}.$

Problem 13P.7 (10 points):

Find the QR decomposition of the following matrices

a) (2 points) $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$

b) (2 points) $B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \\ 3 & 1 \\ 1 & -3 \end{bmatrix}.$

c) (2 points) $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$

d) (2 points) $D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$

e) (2 points) $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Solution:

a) $QR = \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & \frac{2}{3} \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \frac{\sqrt{5}}{3} \end{bmatrix}.$

b) $QR = \begin{bmatrix} \frac{1}{2} & -\frac{3\sqrt{\frac{3}{85}}}{4} \\ \frac{1}{4} & \frac{31^2}{4\sqrt{255}} \\ \frac{1}{4} & \frac{\sqrt{\frac{15}{17}}}{4} \\ \frac{1}{4} & \frac{4}{13} \\ \frac{1}{4} & \frac{4\sqrt{255}}{49} \\ \frac{1}{4} & -\frac{4\sqrt{255}}{49} \end{bmatrix} \begin{bmatrix} 4 & \frac{1}{4} \\ 0 & \frac{\sqrt{255}}{4} \end{bmatrix}.$

c) $QR = 1A$ because A is already upper triangulard) $QR = (A/\sqrt{2})(\sqrt{2}1).$ e) $QR = 1 \cdot 1.$ Both Q and R are the identity

Problem 13P.8 (10 points):

a) (2 points) Find the determinant of the “prime” matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 13 & 0 & 0 & 11 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} .$$

b) (2 points) Find the determinant of the “count to 12” matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix} .$$

c) (2 points) Find the determinant of the “2-1” matrix

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} .$$

d) (2 points) Find the determinant of the “Pascal triangle” matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} .$$

e) (2 points) Find the determinant of the “mystery” matrix:

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 1 & 5 & 5 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix} .$$

Solution:

a) There is just one pattern with three upcrossing: $-13 \cdot 7 \cdot 5 \cdot 3 = -1365$. b) Partitioning the matrix reveals two 2×2 blocks. The determinant is the product of the determinants of the blocks. $(1 \cdot 6 - 2 \cdot 5)(9 \cdot 12 - 11 \cdot 10) = 8$. c) -1 . d) There is only one pattern. The number of upcrossings is 10 so, the determinant is 1. e) Row reduction. Subtract the first

row from all others gives $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$. This matrix has determinant $5! = 120$.

Problem 13P.9 (10 points):

Find the function

$$a|x| - b|x - 1| = y$$

which is the best fit for the data

x	y
1	2
3	2
-2	1
2	0

Solution:

Write the data as a system of equations $Ax = b$ with $A = \begin{bmatrix} 1 & 0 \\ 3 & -2 \\ 2 & -3 \\ 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

The rest is routine. We have $x = (A^T A)^{-1} A^T b = [10, -7]^T / 6$. A few check points: $(A^T A) = \begin{bmatrix} 18 & -14 \\ -14 & 14 \end{bmatrix}$. The inverse is $\begin{bmatrix} 14 & 14 \\ 14 & 18 \end{bmatrix} / 56$. $A^T b = [10, -7]^T$. The best fit is $y = (3/4)|x| - (1/4)|x - 1|$.

Problem 13P.10 (10 points):

The matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

are called the **Gamma matrices** or **Dirac matrices**.

a) (4 points) Are the columns orthonormal? That is, is it true that $A^*A = 1$, where $A^* = \overline{A}^T$.

	A	B	C	D
Orthonormal columns?				

b) (3 points) Compute A^2 and D^2 and $AD + DA$.

c) (3 points) Write down the inverse of A, B and D .

Solution:

These were just routine computations. But they show something interesting about the anticommutation of the matrices. In physics they are written as γ_j and the commutator is written as $\{A, B\} = AB + BA$. What actually happens is that $\{\gamma_j, \gamma_k\} = 2g_{jk}$, where $g = \text{diag}(1, -1, -1, -1)$ is the Lorentz metric. a) All are.

b) $A^2 = 1$, the identity matrix. $D^2 = -1$.

$AD + DA = 0$, the zero matrix

c) $A^{-1} = 1$, $B^{-1} = -B$. $D^{-1} = -D$.