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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 13: First Hourly

Welcome to the first hourly. Please don't get started yet. We start all together at 9:00 AM. You can already fill out your name in the box above. Then relax at the beautiful pond (a Povray scene using code of Gilles Tran who wrote this in 2004).

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. Except when stated otherwise, we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page and at the end.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it. Make sure we find additional work.
- You have 75 minutes for this hourly.

Problems

Problem 13.1 (10 points):
Prove that $\ker(A) = \ker(A^T A)$ for any matrix A. We have seen this to be useful in the data fitting part.

Solution:
a) If x is in the kernel of A, then $Ax = 0$. This implies that $A^T(Ax) = 0$ so that x is in the kernel of $A^T A$.
b) If x is in the kernel of $A^T A$, then $A^T Ax = 0$. This implies that Ax is in $\ker(A^T)$ meaning that it is perpendicular to the image of A. But since Ax is also in the image of A, we have $Ax = 0$ and so x is in the kernel of A. P.S. A couple of students provided another elegant proof for the b): Multiply the equation $A^T Ax = 0$ from the left with x^T. This gives $x^T A^T Ax = 0$ which can be read as $(Ax)^T(Ax) = 0$. This means $(Ax) \cdot (Ax) = |Ax|^2 = 0$ meaning that $Ax = 0$.

Problem 13.2 (10 points) Each problem is 1 point:
Decide in each case whether the set X is a linear space. If it is, prove it, if it is not, then give a reason why it is not.

a) The space of upper triangular 4×4 matrices with zero trace.

b) The space of 2×2 rotation dilation matrices.

c) The space of 2×2 diagonal matrices.

d) The space of 4×4 matrices which have all zeros in the diagonal.

e) The space of all 3×3 matrices with non-trivial kernel.

f) The kernel of $A^T A$, where A is a 10×3 matrix.

g) The 3×3 matrices which row reduce to the 1 matrix.

h) The 3×3 matrices A which have the property that $e^A = 1$.

i) The 2×2 rotation dilation matrices zero trace.

j) The set of 4×4 quaternion matrices satisfying $A^2 = -1$.

Solution:
a) yes
b) yes
c) yes
d) yes
e) no, take two projection matrices which add up to the identity.
f) yes
g) no, zero is not in.
h) yes. It is however not trivial to see that $A = 1$ is the only solution to this. We will learn about this later. We counted also no as a correct answer here as there was no theory yet to back yes up.
i) yes
j) no. 0 is not in.

Problem 13.3 (10 points):
Decide in each case, whether the transformation T is linear. If it is, prove it, if it is not, then give a reason why it is not.

a) (1 point) $T(A) = A \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, from $M(2,2)$ to $M(2,2)$.

b) (1 point) $T(A) = A^2$ from $M(2,2)$ to $M(2,2)$.

c) (8 points) Match each of matrices with one of the geometric descriptions below. You don't have to give explanations in this part c)

Matrix	Enter A-H here.	Matrix	Enter A-H here.
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	
$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$		$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$		$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

A) Shear along a plane.
B) Projection onto a plane.
C) Rotation around an axes.
D) Reflection about a point.
E) Projection onto a line.
F) Reflection about a plane.
G) Reflection about a line.
H) Identity transformation.

Solution:
a) yes, b) no
B,A
C H
D E
F G

Problem 13.4 (10 points, each sub problem is 2 points):
a) Who was the inventor of quaternions?

b) The SU in $SU(2)$ stands for Super dUper. No seriously, what does SU abbreviate?

c) What can you say about the determinant of an orthogonal matrix?

d) What are the properties of a monoid?

e) Is it possible to have a 3×5 matrix with orthonormal columns? If yes, give one.

Solution:
a) Hamilton
b) Special Unitary
c) ± 1 d) Associativity and 0 element
e) no, there are at least 2 free variables and so a kernel.

Problem 13.5 (10 points):
a) (6 points) Find a basis for the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

b) (4 points) Find a basis for the image of A.

Problem 13.6 (10 points):

a) (5 points) What are the coordinates of the vector $\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$ in the basis

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \} ?$$

b) (5 points) A transformation T is described in the standard basis by

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$

What is the matrix B of the transformation T in the basis $\mathcal{B}$?

Solution:

a) The matrix S which contains the basis vectors as columns is

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

Its inverse is

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} /2 .$$

The vector $\vec{v}$ in the $\mathcal{B}$ coordinates is $[\vec{v}]_{\mathcal{B}} = S^{-1}v = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

b) The matrix B in the basis $\mathcal{B}$ is

$$B = S^{-1}AS = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} /2 = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix} /2 .$$

Problem 13.7 (10 points):
a) (2 points) Find the QR Decomposition of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & -4 \\ 4 & 3 \end{bmatrix}$$

b) (2 points) Find the QR decomposition of the product

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} .$$

c) (2 points) Find the QR decomposition of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} .$$

d) (2 points) Find the QR decomposition of

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

e) (2 points) Find the QR decomposition of the 1×1 matrix

$$A = [5] .$$

Solution:
They are all quite obvious.
a) $Q = A/\sqrt{30}$, $R = \sqrt{30} 1_2$
b) $A = QR$ is already decomposed.
c) $Q = A/\sqrt{5}$, $R = \sqrt{5} 1_2$.
d) $Q = A/\sqrt{30}$, $R = \sqrt{30} 1_1$
e) $Q = 1, R = 5$.

Problem 13.8 (10 points):
Make sure to indicate which method you use to compute the determinant!

a) (2 points) Find the determinant $A = \begin{bmatrix} 1 & 2 & 4 & 8 & 2 \\ 0 & 0 & 3 & 4 & 3 \\ 7 & 0 & 0 & 0 & 4 \\ 9 & 0 & 0 & 7 & 3 \\ 6 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) (2 points) Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

c) (2 points) Find the determinant $A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 1 & 3 \\ 2 & 2 & 0 & 4 & 2 \end{bmatrix}$

d) (2 points) Find the determinant $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

e) (2 points) Find the determinant $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 6 & 8 & 10 \\ 1 & 2 & 6 & 8 & 13 \\ 1 & 2 & 3 & 8 & 10 \\ 1 & 2 & 3 & 4 & 9 \end{bmatrix}$

Solution:
a) -1008 one pattern with 5 upcrossings
b) One row reduction or two patterns $1 * 2 * 3(14-1) = 6 * 13 = 78$.
c) Row reduce -2
d) Partitioned: -240
e) Row reduce to triangular. 96.

Problem 13.9 (10 points):
Find the equation of the form

$$x^2 + axy + by^2 = 1$$

that best fits the data points:

x	y
2	1
-1	1
1	0
0	1

Solution:

Writing down the equations and simplifying gives $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. We have $A^TA = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$ which has the inverse $(A^TA)^{-1} = \begin{bmatrix} \frac{3}{14} & -\frac{1}{14} \\ -\frac{1}{14} & \frac{5}{14} \end{bmatrix}$. Also, $A^Tb = [-6, -2]^T$. Now $(A^TA)^{-1}A^Tb = \begin{bmatrix} -\frac{8}{7} \\ -\frac{2}{7} \end{bmatrix}$. The best solution is $x^2 - (8/7)xy - (2/7)x^2$.

Problem 13.10 (10 points) every entry is 1 point:
Match the following matrices with the correct label. No justifications are needed. Fill in the attribute acronym into the middle boxes. One attribute might apply to more than one. But there is a unique match which works for all. To the right, indicate whether the transformation is invertible or not.

	Transformation matrix	attribute	invertible (yes or no)?
A=	$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix}$		
B=	$\begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$		
C=	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$		
D=	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
E=	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$		

Here are the attribute acronyms.

ANTI) anti-symmetric matrix

REFL) reflection matrix

PROJ) orthogonal projection

SYMM) symmetric matrix

ORTH) orthogonal matrix

Solution:
SYMM Yes
ANTI No
ORTH Yes
PROJ No
REFL Yes