Unit 14: Characteristic Polynomial

Lecture

14.1. We have seen the definition of determinants as well as methods to compute them. In this lecture, we will learn another method which is based on the **fundamental theorem of algebra**.

14.2. If A is a $n \times n$ matrix and v is a non-zero vector such that $Av = \lambda v$, then v is called an **eigenvector** of A and λ is called an **eigenvalue**. We see that v is an eigenvector if it is in the kernel of the matrix $A - \lambda 1$. We know that this matrix has a non-trivial kernel if and only if $p(\lambda) = \det(A - \lambda 1)$ is zero. By the definition of determinants the function $p(\lambda)$ is a polynomial of degree n.

14.3. In order to study the **characteristic polynomial**

$$p_A(\lambda) = \det(A - \lambda 1)$$

we first of all need to know the **fundamental theorem of algebra**:

Theorem: A polynomial $f(x)$ of degree n has exactly n roots in $\mathbb{C}$.

The roots are counted with multiplicity. $f(x) = x^2 + 2x + 1$ for example has two roots $-1, -1$ so that it factors as $f(x) = (x + 1)(x + 1)$. The roots can be complex like $f(x) = x^2 + 1$, which factors as $(x - i)(x + i)$. The theorem implies

$$p_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) .$$

It is enough to prove that one root λ_1 exists. By induction we can then factor the degree $n - 1$ polynomial $p(\lambda)/(\lambda - \lambda_1)$.

Proof. The Bolzano extremal value theorem assures that a continuous function f on a closed disk has a minimum and maximum. This implies that the function $|f(z)|$ has a global minimum in $\mathbb{C}$. (There exists r_0 such that the minimum of $|f(z)|$ with $|z| = r$ is larger than say $f(0)$ for $r \geq r_0$ so that the minimum has to be in the disk $\{|z| \leq r_0\}$.) We use the method of contradiction to show that the minimal value $|f(z_0)|$ is zero. Assume $f(z_0) \neq 0$ allows to introduce $g(z) = f(z_0 + z)/f(z_0)$ which is also a polynomial of degree n with minimum 1 at z_0. We have $g(z) = 1 + b_k z^k + \cdots + b_n z^n$ with $b_k = |b_k| e^{i\theta} \neq 0$ for some k. Define the line $z(t) = t|b_k|^{-1/k} e^{i(\pi - \theta)/k}$. Then $g(z(t)) = 1 - t^k + t^{k+1} h(t)$ with some continuous function $h(t)$. The value of $|g(z(t))| \leq 1 - t^k + t^{k+1}|h(t)|$ is less than 1 for small positive t contradicting that g had a minimal value 1 at z_0. $\square$

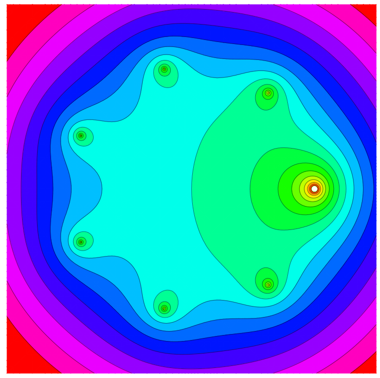

FIGURE 1. Contour map $(x+iy) \to |p(x+iy)|$ of the characteristic polynomial of the 8-queen matrix. Here, from the 8 roots, $\lambda = 1$ appear with multiplicity 2. (The algebraic multiplicity of 1 is 2.)

14.4. This implies

The determinant of A is the product of the eigenvalues of A.

The trace of A is the sum of the eigenvalues of A.

14.5. This gives us a new way to compute determinants. Find the eigenvalues and take the product of the eigenvalues. Example: to find the determinant of

$$A = \begin{bmatrix} 11 & 1 & 1 & 1 & 1 \\ 1 & 11 & 1 & 1 & 1 \\ 1 & 1 & 11 & 1 & 1 \\ 1 & 1 & 1 & 11 & 1 \\ 1 & 1 & 1 & 1 & 11 \end{bmatrix}$$

define $B = A - 10I$ which has a 4 dimensional kernel and so 4 eigenvalues 0. The last eigenvalue of B is 5 as the sum of the eigenvalues is $tr(B) = 5$. Since B has the eigenvalues $0, 0, 0, 0, 5$, the matrix A has the eigenvalues $10, 10, 10, 10, 15$. The determinant is 150000. We can even write down the characteristic polynomial

$$p_A(\lambda) = (\lambda - 10)^4(\lambda - 15) .$$

14.6. We are interested in the coefficients of the characteristic polynomial.

The polynomial starts with $(-\lambda)^n$ so that $a_n = (-1)^n$.

The coefficient $(-1)^{n-1}a_{n-1}$ is the trace of A.

The coefficient a_0 is the determinant of A.

14.7. Problem: Find $p_Q(\lambda$ for the **magic square**: $A = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$ and factor it.

Solution. Just compute the determinant

$$\det(A - \lambda) = \det(\begin{bmatrix} 4-\lambda & 9 & 2 \\ 3 & 5-\lambda & 7 \\ 8 & 1 & 6-\lambda \end{bmatrix}) = -\lambda^3 + 15\lambda^2 - 24\lambda + 360 \ .$$

We double check that a_0 is the determinant of A. The start is always $(-\lambda)^n$. The term in from of λ^2 is the trace of A. The only coefficient which was not so easy to get is the -24λ. How do we factor the polynomial? We know that $\lambda = 15$ is a root because $[1,1,1]^T$ is an eigenvector as it is the sum of the row elements. So, we can divide $(-\lambda^3 + 15\lambda^2 - 24\lambda + 360)/(15 - \lambda)$ which is $24 + x^2$. So, we have $p_A(\lambda) = (15 - \lambda)(i\sqrt{24} - \lambda)(-i\sqrt{24} - \lambda)$.

Problem: Find the characteristic polynomial of the **Boolean matrix** encoding one of the solutions of the **8 queen problems**

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Solution. The matrix A has only one nonzero pattern. The determinant is 1. The trace is 1. So, we know the polynomial looks like $\lambda^8 - \lambda^7 + ... + 1$. The matrix $A - \lambda 1$ is partitioned with a 1×1 and 7×7 matrix. The characteristic polynomial of the 7×7 matrix is $(-\lambda^7 + 1)$. The characteristic polynomial of the 1×1 matrix is $1 - \lambda$. We get (see homework 14.3) $(1 - \lambda)(1 - \lambda^7) = \lambda^8 - \lambda^7 - \lambda + 1$.

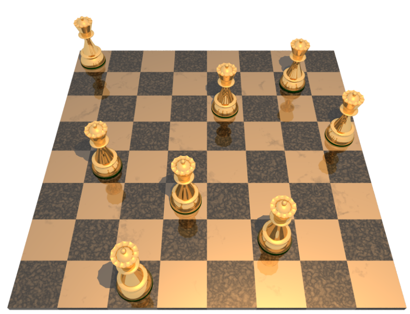

Figure 2. A solution to the 8 queen problem

Homework

This homework is due on Tuesday, 3/5/2019.

Problem 14.1: Find $p(\lambda)$ and factor: a) $A = 4$, b) $B = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$, c) $C = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, d) $D = \begin{bmatrix} 11 & 1 & 1 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$, e) $E = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.

Problem 14.2: Find the determinant and characteristic polynomial:

$$A = \begin{bmatrix} 101 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 102 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 103 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 104 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 105 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 106 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 107 \end{bmatrix}$$

Problem 14.3: a) Express the characteristic polynomial of the **partitioned matrix** det $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ in the form of the characteristic polynomials of A and B.
b) What is the relation between the $p_{A^T}(\lambda)$ and $p_A(\lambda)$?

Problem 14.4: a) Given the eigenvalues $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 4, \lambda_4 = 9$, find a non-triangular matrix which has these eigenvalues.
b) Given the eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 3 + 4i$, $\lambda_3 = 2 - 2i$ is there a real matrix 3×3 matrix which has these eigenvalues? If no, why not?
c) Is it true that if λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB?
d) Is it true that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.
e) True or False: if λ is a non-zero eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.
f) True or False: if λ is an eigenvalue of A, then λ is an eigenvalue of A^T.

Problem 14.5: There are sometimes fast ways to compute the characteristic polynomials: An example

$$A = \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 8 \\ 3 & 3 & 6 \end{bmatrix} .$$

You can virtually "see" the characteristic polynomial. How?